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INDEX

PAGE

ALEXANDER, J. W., II, Normal Forms for One- and Two-Sided Surfaces... 158
BENNETT, A. A., An Algebraic Treatment of the Theorem of Closure..... 97
BLISS, G. A., A Note on Symmetric Matrices..... 43
BLISS, G. A., A Substitute for Duhamel's Theorem..... 45
BROWN, T. H., The Effect of Radiation on a Small Particle Revolving about Jupiter..... 22
COPELAND, L. P., On the Theory of n-Lines..... 7
CRAIG, C. F. See Sharpe..... 15
DICKSON, L. E., The Points of Inflexion of a Plane Cubic Curve..... 50
DICKSON, L. E., Projective Classification of Cubic Surfaces Modulo 2..... 139
EMCH, A., On Some General Theorems Concerning Ordinary Closed Curves. 193
FISCHER, C. A., Minima of Double Integrals with Respect to One-Sided Variations..... 162
GALAJIKIAN, H., Non-Linear Integral Equations of the Volterra Type..... 172
GRONWALL, T. H., Some Remarks on Conformal Representation..... 72
GRONWALL, T. H., On the Maximum Modulus of an Analytic Function... 77
GRONWALL, T. H., An Integral Equation of the Volterra Type..... 119
HAZLETT, O. C., Invariantive Characterization of Some Linear Associative Algebras..... 1
HEBBERT, C. M., The Inscribed and Circumscribed Squares of a Quadrilateral and their Significance in Kinematic Geometry..... 38
HEBBERT, C. M., Properties of Four Confocal Parabolas Whose Vertical Tangents Form a Square..... 67
IRWIN, F., Relation Between the Roots of a Rational Integral Function and its Derivative..... 138
MILLER, G. A., A New Proof of Sylow's Theorem..... 169
MOORE, C. L. E., Note on Normal Sections of a Surface in a Space of n Dimensions..... 89
MOORE, R. L., The Linear Continuum in Terms of Point and Limit..... 123
MORGAN, F. M., A Plane Cubic Cremona Transformation and its Inverse.. 134
PELL, A. J., Non-Homogeneous Linear Equations in Infinitely Many Unknowns..... 32
SHARPE, F. H., and CRAIG, C. F., Plane Curves with Consecutive Double Points..... 15
WEDDERBURN, J. H. M., Note on the Simple Difference Equation..... 82
WEDDERBURN, J. H. M., Note on the Rank of a Symmetrical Matrix..... 86
ERRATA.....88, 138

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INVARIANTIVE CHARACTERIZATION OF SOME LINEAR ASSOCIATIVE ALGEBRAS.

By Olive C. Hazlett.*

Linear associative algebras of a small number of units, with coördinates in the field of ordinary complex numbers, have been completely tabulated,† and their multiplication tables have been reduced to very simple forms. But if we had before us a linear associative algebra, the chances are that its multiplication table would not be in such a form that we could find out readily to what standard form it was equivalent. And so the question arises, " May we not find invariants which completely characterize these algebras? "

We use the term invariant here in the sense defined by Professor Dickson, —that is, a single-valued function (in Dirichlet's sense) of the constants of multiplication which takes on the same value for two algebras of the same number of units (having constants of multiplication and coördinates in any field F) if the two algebras be equivalent with respect to F,—that is, if there be a non-singular linear homogeneous transformation‡ with coefficients in F which will transform the units of one algebra into the units of the second algebra. The invariants $I_1, \cdots, I_m$ are said to completely characterize the algebras of n units over a field F when each I_k has the same value for two algebras only when they are equivalent.

It turns out that, in the field C of ordinary complex numbers, under the general group T of linear transformations, the binary and ternary linear associative algebras with a modulus are both completely characterized by the rank of the algebra and by the rank of certain homogeneous quadratics in the constants of multiplication and the coördinates of the general number of the algebra.

In order to find invariants in the more technical sense defined above, we will first find some rational functions which, in the more usual sense, remain invariant under any linear transformation on the units. We will mean by a *rational invariant* of the n-ary linear algebra $(e_1, \cdots, e_n)$ with

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† See especially B. Peirce: "Linear Associative Algebras," Am. Jour. of Math., vol. 4 (1881), p. 97–192; G. Scheffers: "Zurückführung complexer Zahlensysteme auf typische Formen," Math. Ann., vol. 39 (1891) p. 293–390; Study: "Complexen Zahlen," Gött. Nach., 1889, p. 237–268.

‡ Hereafter in this paper, the word transformation will be used to denote a non-singular linear homogeneous transformation.

coördinates in F and indeterminate constants of multiplication γ_{ijk} $(i, j, k = 1, \cdots, n)$, a rational function of the γ's which, when the units of the algebra are subjected to any linear transformation with coefficients in F, is merely multiplied by some power of the determinant of the transformation. *Rational covariants* are defined in a similar manner.

1. Rational Invariants of General Linear Algebras.—Both characteristic determinants*

$$\delta \equiv |\sum_i \gamma_{ijk}x_i - d_{jk}\omega|, \qquad \delta' \equiv |\sum_i \gamma_{jik}x_i - d_{jk}\omega| \quad (j, k = 1, \cdots, n)$$

are rational covariants and thus produce invariants. For the right-hand determinant δ is the determinant of the n linear homogeneous equations

$$\sum_{i,j=1}^{n} (\gamma_{ijk}x_i - d_{jk}\omega)y_j = 0 \qquad (k = 1, \cdots, n),$$

which are the necessary and sufficient conditions that it be possible to find a scalar ω and a number $Y = \Sigma_i y_i e_i$ such that $XY = \omega Y$.† But a necessary and sufficient condition that there be a solution $Y \neq 0$ is that ω be a root of $\delta = 0$. Suppose ω_0 is such a root, or $XY = \omega_0 Y$. If the units of the algebra be subjected to a non-singular linear transformation (with coefficients in F) such that x_i goes into X_i and γ_{ijk} in Γ_{ijk}, the two quantities X and Y are unchanged and thus ω_0 is unchanged. But ω_0 must be a root of $\delta(X_i, \Gamma_{ijk})$ by the foregoing, and hence $\delta(X_i, \Gamma_{ijk})$ vanishes with $\delta(x_i, \gamma_{ijk})$.

Moreover δ is an irreducible function of ω where the constants of multiplication are indeterminates. Suppose $\delta = \delta_1\delta_2$. No one γ_{ijk} can occur in both δ_1 and δ_2; and moreover if δ be actually reducible δ_1 and δ_2 must each contain γ's of the form γ_{ijj}. Suppose γ_{i11} be in δ_1; then γ_{j1k} and γ_{jk1} cannot occur in δ_2 $(j, k = 1, \cdots, n)$. Similarly, if γ_{ill} be one of the γ's which actually occur in δ_2, γ_{jlk} and γ_{jkl} $(j, k = 1, \cdots, n)$ cannot occur in δ_1. Or γ_{j1l} and γ_{jl1} $(j = 1, \cdots, n)$ cannot occur in δ_1 or δ_2, which is impossible.

But, if a polynomial φ be irreducible and if a polynomial ψ vanish whenever φ vanishes, then φ is a factor of ψ. Consequently, since the coefficient of ω^n is $(-1)^n$ in δ formed for both the original and transformed algebra, δ is an absolute covariant. Similarly with δ'. Furthermore, since ω is an arbitrary variable, the coefficients of the powers of ω in δ and δ' must separately be absolute covariants.

If we write $\delta = \Sigma_{i=0}^{n} C_i\omega^i$ and $\delta' = \Sigma_{i=0}^{n} C_i'\omega^i$, then the C's and C''s are homogeneous polynomials in the coördinates, with coefficients homogeneous in the γ's. Now these polynomials produce invariants of the algebra.

* ω is an arbitrary variable which may take on values in field F or in the algebra; and the n x_i's are the coördinates of any number in the algebra.

† Cartan: Annales de Toulouse, séries I, vol. 12 (1898); B. 16.

We will mean by a rational covariant of a covariant K of the n-ary linear algebras with coördinates in the field F and undetermined γ's, a rational function K' of the coördinates and of the coefficients of K which, when the coördinates are subjected to a linear transformation (with coefficients in F), is merely multiplied by some power of the determinant of the transformation. If K' do not contain the coördinates, it will be called a rational invariant of K. Just as, in the invariant theory for quantics, any covariant or invariant of a covariant of a quantic is itself a covariant or invariant of the quantic, so any covariant or invariant of a covariant of a linear algebra is itself a covariant or invariant of the algebra. Hence we have

THEOREM I. *The rational, integral invariants and covariants of the characteristic determinants and of the coefficients of the powers of ω therein are respectively invariants and covariants of the n-ary linear algebras with coördinates and constants of multiplication in any field.*

2. Ternary Linear Associative Algebras with a Modulus.—We are now in a position, from Study's results,* to exhibit some invariants which completely characterize linear associative algebras of two and three units having a modulus.† In this article, Study separates into types all linear associative algebras of two, three and four units having a modulus, according to the rank and according to the character of the rank equation as shown by its number of multiple roots and their multiplicities. He takes as his field F the field C of all complex numbers. His results for $n = 3$ follow.

$r = 3$‡

I

	e_1	e_2	e_3
e_1	e_1	0	0
e_2	0	e_2	0
e_3	0	0	e_3

II

	e_1	e_2	e_3
e_1	e_1	e_2	0
e_2	e_2	0	0
e_3	0	0	e_3

III

	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_3	0
e_3	e_3	0	0

$\delta = \delta' = \prod_{i=1}^{3}(x_i - \omega)$, $\delta = \delta' = (x_1 - \omega)^2(x_3 - \omega)$, $\delta = \delta' = (x_1 - \omega)^3$.

* See note 1, page 1.

† That is, a number E having the properties of ordinary unity.

‡ $r = 3$ means that three is the degree of the equation of lowest degree satisfied by every number in the algebra; or that the rank is three. $r = 2$ means the rank is two. Clearly the rank is an invariant in the more technical sense.

$r = 2$*

IV

	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	e_1	e_3
e_3	e_3	$-e_3$	0

$\delta = (x_1 + x_2 - \omega)(\omega^2 - 2x_1\omega + x_1^2 - x_2^2),$
$\delta' = (x_1 - x_2 - \omega)(\omega^2 - 2x_1\omega + x_1^2 - x_2^2),$

V

	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	0	0
e_3	e_3	0	0

$\delta = \delta' = (x_1 - \omega)^3.$

Since the rank of a quadratic form is invariant, and since for two algebras of the same rank but of different types, the rank of C_1 is different, we see that

THEOREM II. *In the field of all complex numbers the ternary linear associative algebras with a modulus are completely characterized by their rank and by the rank of the coefficient of ω in either characteristic determinant.*

Type	Rank	Rank of C_1
I	3	3
II	3	2
III	3	1
IV	2	2
V	2	1

3. General Binary Linear Associative Algebras.—The invariants which characterize the binary linear associative algebras with a modulus can be obtained in a similar manner from Study's results for the binary algebras; but we can do better—by reduction from the ternary algebras with a modulus, we can find the invariants which characterize all binary linear associative algebras (with or without a modulus).

An n-ary linear algebra $(e_1, \cdots, e_n)$ with coördinates and constants of multiplication in a field F is said to be *reduced* when in every number of this algebra we set $x_{m+1} = \cdots = x_n = 0$. The system thus defined may or may not be closed,—i. e. the product of any two numbers in the system may or may not be in the system. If such products always remain in the system, and if the coördinates of $(e_1, \cdots, e_m)$ range over F, the numbers are said to form a *subalgebra* of the original algebra.

In particular, take the ternary linear algebras (not necessarily associative) over a field F where e_3 is the modulus, and reduce by setting $x_3 = 0$;

* Study says that, in this case "we will choose two units beside $e_1 = 1$ such that either $e_2^2 = e_3^2 = e_1$ or $e_2^2 = e_3^2 = 0$"; but conceivably there might be an algebra in which we could not do this,—where at best we could choose units such that $e_2^2 = e_1$, $e_3^2 = 0$. Such however is readily shown impossible for $r = 2$.

and consider the resulting binary algebras. For convenience, two right-hand (or two left-hand) characteristic determinants for binary algebras over a field F will be said to be of the same type if the discriminants of their constant terms be the same except for a non-zero factor which is a perfect square in F; and if, when the coefficient of ω vanishes identically in one determinant, it does in the other also. Since the invariants which characterize the initial algebra necessarily characterize the sub-algebras, no two non-equivalent ternary linear algebras can, by this reduction, produce two equivalent binary algebras. Finally, since by adjoining a modulus to a binary algebra, we have a ternary algebra with a modulus, these are all the binary linear algebras. Thus the binary algebras are partially characterized by the type of their characteristic determinants and by any set of invariants which completely characterize the algebras with three units. But how far do these characterize them? how many non-equivalent binary algebras are produced by any one type of ternary algebras?

If two subalgebras $a : (e_1, e_2)$ and $a' : (e_1', e_2')$ obtained by this reduction from equivalent ternary algebras are not equivalent, then one or both units of a' are linearly independent of the units of a. In the second case, since there are only three linearly independent numbers in the original algebra, the units of a' can be so transformed that we have the first case.

Now this first case is possible only if the two algebras can be transformed such that their units are respectively of the forms (e_1, e_2) and $(e_1' = E - e_1, e_2)$ where E is the modulus of the original algebra. For E must be expressible as the sum of two numbers, one in a, the other in a'. Suppose $e_2' = b_1e_1 + b_2e_2$, $E = \epsilon_1e_1 + \epsilon_2e_2 + \epsilon_1'e_1' + \epsilon_2'e_2'$. If in a, we set $E_1 = \epsilon_1e_1 + \epsilon_2e_2$, $E_2 = b_1e_1 + b_2e_2$, the transformation is non-singular. For otherwise E_1 would be a scalar multiple of e_2' and thence E would be in a', which is impossible. Similarly, since $E - E_1$ is linearly independent of e_2', we can set $E_1' = E - E_1$ and $E_2' = e_2'$ and the algebras are in desired form.

From the identity $e_1'e_2' = e_2 - e_1e_2$ we infer that $e_1e_2 = me_2$; and similar arguments for the other products show that the multiplication tables for these two algebras are of the forms

a:

	e_1	e_2
e_1	$e_1 + he_2$	me_2
e_2	ne_2	le_2

a':

	e_1'	e_2'
e_1'	$e_1' + he_2'$	$(1 - m)e_2'$
e_2'	$(1 - n)e_2'$	le_2'

Thus there are at most two non-equivalent binary algebras produced from a ternary algebra by the above reduction. Moreover the right-hand characteristic determinants for these algebras are not of the same type, for they

are

a: $\quad \delta = \omega^2 - \omega[(m+1)x_1 + lx_2] + x_1[mx_1 + lx_2],$

a': $\quad \delta = \omega^2 - \omega[(2-m)x_1 + lx_2] + x_1[(1-m)x_1 + lx_2].$

Hence, by theorem I and the argument at the beginning of this section, we have

THEOREM III. *The binary linear algebras over any field F are characterized completely by the type of their characteristic determinants and by any set of invariants which completely characterizes the ternary algebras with a modulus.*

The binary linear associative algebras over the field of all real and complex numbers are as follows:

I

	e_1	e_2
e_1	e_1	0
e_2	0	e_2

$\delta = \delta' = (x_1 - \omega)(x_2 - \omega),$

II

	e_1	e_2
e_1	e_1	e_2
e_2	e_2	0

$\delta = \delta' = (x_1 - \omega)^2,$

III

	e_1	e_2
e_1	e_1	0
e_2	0	0

$\delta = \delta' = -\omega(x_1 - \omega),$

IV

	e_1	e_2
e_1	e_2	0
e_2	0	0

$\delta = \delta' = \omega^2,$

V

	e_1	e_2
e_1	0	e_1
e_2	0	e_2

$\delta = -\omega(x_2 - \omega),$
$\delta' = (x_2 - \omega)^2,$

VI

	e_1	e_2
e_1	e_1	e_2
e_2	0	0

$\delta = (x_1 - \omega)^2,$
$\delta' = -\omega(x_1 - \omega),$

VII

	e_1	e_2
e_1	0	0
e_2	0	0

$\delta = \delta' = \omega^2.$

Of these, I, IV and VII are obtained by reduction from types I, III and V respectively in Study's list of ternary algebras; whereas II and III are given by type II of these ternary algebras, and V and VI by type IV.* All algebras but the last are completely characterized by the type of the characteristic determinants; and thus we have

THEOREM IV. *The non-trivial linear associative binary algebras are completely characterized by the rank of C_0 and C_0', and by the identical vanishing or non-vanishing of C_1.*

* Peirce (loc. cit., pp. 120–122) says there are only three irreducible (or pure) linear associative algebras of two units and one irreducible (or mixed) algebra. The three algebras he tabulates are of types II, IV and VI in the above list. Cayley, I find, points out this error in "Double Algebra," Proceedings London Math. Soc., vol. XV (1884); Collected Works, XII, p. 60. He here gives all the seven types of binary associative algebras. Like Peirce, he gets them directly and not by reduction.

ON THE THEORY OF INVARIANTS OF n-LINES.

By Lennie Phoebe Copeland.

The invariants of plane polygons or n-lines have been studied from various points of view by Morley,* Leib,† Hun,‡ Glenn,§ Loud|| and MacNeish.¶

The purpose of this paper is to construct the elements of the formal theory of the plane m-line, where the latter is assumed to be represented always by a ternary form whose coefficients are given in terms of the absolute minimum number of independent parameters. For a ternary form f_{3m} to represent an m-line this minimum number is $2m$ and all the coefficients of f_{3m} are then rational in the parameters.

In fact it has been shown by Glenn that if the $2m$ parameters be taken as the coefficients of $a_{0x}{}^m$, $a_{1x}{}^{m-1}$, where

$$f_{3m} = a_{0x}{}^m + x_3 a_{1x}{}^{m-1} + \cdots + x_3{}^m a_{m0}$$

and

$$a_{ix}{}^{m-i} \equiv a_{i0}x_1{}^{m-i} + a_{i1}x_1{}^{m-i-1}x_2 + a_{i2}x_1{}^{m-i-2}x_2{}^2 + \cdots + a_{im-i}x_2{}^{m-i}$$

$$(i = 0, 1, 2, \cdots m),$$

then the remaining coefficients of f_{3m} are expressible as simple rational invariants of $a_{0x}{}^m$, $a_{1x}{}^{m-1}$. In section (1) we consider seminvariantive sets of conditions in order that two 2-lines, or two 3-lines may be projectively related to a triangle, by certain apolar laws. In section (2) we study the theory of full invariants and complete systems of m-lines. Applications of this theory are given in section (3).

The present writer wishes to thank Professor O. E. Glenn, who suggested the undertaking of this investigation, for numerous helpful suggestions and criticisms.

* Morley, "On the metric properties of the plane n-line," Am. Math. Soc. Trans., vol. 1; "On the orthocentric properties of the plane n-line," Am. Math. Soc. Trans., vol. 4, 5; "On two cubic curves in triangular relations," London Math. Soc. Proceedings (2), 4.

† Leib, "On the invariants of two triangles," Am. Math. Soc. Trans., vol. 10.

‡ Hun, "Invariant relations of two triangles," Am. Math. Soc. Trans., vol. 5.

§ Glenn, "On semi-discriminants of ternary forms," Trans. of the Am. Math. Soc., vol. 12, pp. 367–374; "On the structure of forms and the algebraical theory of n-lines," Am. Journal of Mathematics, vol. 34, No. 4.

|| Loud, "Sundry metric theorems concerning n-lines in a plane," Am. Math. Soc. Trans., vol. 1.

¶ MacNeish, "Linear polars of k-hedrons in n-space," University of Chicago Press, 1912.

1. Seminvariants of f_{32}, f_{33}. The form of the ternary m-line quantic due to Glenn is

$$f_{3m} = x_2{}^m l_{0\,x_1/x_2} + x_2{}^{m-1} x_3 l_{1\,x_1/x_2} + D^{-1}(-1)^{\frac{1}{2}m(m-1)}$$

$$(1) \qquad \times \sum_{j=2}^{m} x_3{}^j \sum_{i=0}^{m-j} \frac{\Delta_1{}^{m-j-i} \Delta_2{}^i R_m}{(m-j-i)!\, i!}\, x_1{}^{m-j-i}\, x_2{}^i$$

$$\equiv \prod_{i=1}^{m} \left(x_1 + r_i x_2 - \frac{l_{1-r_i}}{l'_{0-r_i}} \right)$$

where

$$\Delta_1 = m a_{00} \frac{\partial}{\partial a_{10}} + (m-1) a_{01} \frac{\partial}{\partial a_{11}} + \cdots + a_{0m-1} \frac{\partial}{\partial a_{1m-1}},$$

$$\Delta_2 = m a_{0m} \frac{\partial}{\partial a_{1m-1}} + (m-1) a_{0m-1} \frac{\partial}{\partial a_{1m-2}} + \cdots + a_{01} \frac{\partial}{\partial a_{10}},$$

$$x_2{}^{m-k} l_{k\,x_1/x_2} = a_{k0} x_1{}^{m-k} + a_{k1} x_1{}^{m-k-1} x_2 + \cdots + a_{km-k} x_2{}^{m-k},$$

and R_m is the resultant of $x_2{}^m l_{0\,x_1/x_2}$ and $x_2{}^{m-1} l_{1\,x_1/x_2}$, and D the discriminant of $x_2{}^m l_{0\,x_1/x_2}$. Consider a pair of 2-lines

$$f_{32} = x_2{}^2 l_{0\,x_1/x_2} + x_2 x_3 l_{1\,x_1/x_2} - \frac{R_2}{D} x_3{}^2 = \alpha_{1x}\alpha_{2x} = 0,$$

$$g_{32} = x_2{}^2 \varphi_{0\,x_1/x_2} + x_2 x_3 \varphi_{1\,x_1/x_2} - \frac{R_2'}{D'} x_3{}^2 = \beta_{1x}\beta_{2x} = 0.$$

These will intersect the sides of a given triangle, chosen as the triangle of reference, in harmonic ranges provided only

$$C_1 = 2a_{00}b_{02} + 2a_{02}b_{00} - a_{01}b_{01} = 0,$$

$$(2) \qquad C_2 = 2a_{00}DR_2' + 2b_{00}D'R_2 + a_{10}b_{10}DD' = 0,$$

$$C_3 = 2a_{02}DR_2' + 2b_{02}D'R_2 + a_{11}b_{11}DD' = 0.$$

We can, by referring to (1) ($m = 3$), also obtain a set of necessary and sufficient conditions in order that two given triangles f_{33}, g_{33} may cut the respective sides of a third triangle (chosen as the triangle of reference) in apolar point ranges. These are

$$C_1' = 3a_{00}b_{03} - a_{01}b_{02} + a_{02}b_{01} - 3a_{03}b_{00} = 0,$$

$$(3) \qquad C_2' = 3a_{00}DR_3' - a_{10}D\Delta_1 R_3' + b_{10}D'\Delta_1 R_3 - 3b_{00}D'R_3 = 0,$$

$$C_3' = 3a_{03}DR_3' - a_{12}D\Delta_2 R_3' + b_{12}D'\Delta_2 R_3 - 3b_{03}D'R_3 = 0.$$

Similar sets of conditions are obtainable for the general case of two m-lines, directly from (1).

2. General Theory. Let the form f_{3m} (see (1)) be written as follows:

$$f_{3m} = \prod_{i=1}^{m} (r_{i1}x_1 + r_{i2}x_2 + r_{i3}x_3),$$

and let

$$a_{k_1k_2k_3} = \Sigma r_{11}r_{21} \cdots r_{k_11}r_{k_1+12}r_{k_1+22} \cdots r_{k_1+k_22}r_{k_1+k_2+13} \cdots r_{k_1+k_2+k_33},$$

where Σ is the ordinary elementary symmetric function of the three groups of homogeneous variables r. Then

$$\delta_{12}a_{k_1k_2k_3} = \left(r_{11}\frac{\partial}{\partial r_{12}} + r_{21}\frac{\partial}{\partial r_{22}} + \cdots + r_{m1}\frac{\partial}{\partial r_{m2}}\right)a_{k_1k_2k_3} = (k_1+1)a_{k_1+1k_2-1k_3}, \tag{4}$$

$$\delta_{23}a_{k_1k_2k_3} = (k_2 + 1)a_{k_1k_2+1k_3-1}, \text{ etc.}$$

If any function of the roots r_{ij}, as $I(r)$, is expressible in terms of the coefficients

$$I(r) = J(a)$$

and if δ_{ij} annihilates I, then, as is readily shown by expanding $I(r)$ by Taylor's theorem,*

$$\Delta_{ij} = \sum_k \frac{\partial}{\partial a_{k_1k_2k_3}}(k_i + 1)a_{\ldots k_i+1, \ldots, k_j-1, \ldots} \qquad (i, j = 1, 2, 3, i \neq j) \tag{5}$$

annihilates $J(a)$.

We proceed to show that any function $I(r)$ which has the annihilators δ_{ij} is a function of the determinants of the third order that can be formed from triads of the factors of f_{3m}.†

We assume

$$\delta_{12}I = \delta_{13}I = \delta_{21}I = \delta_{23}I = \delta_{31}I = \delta_{32}I = 0. \tag{6}$$

From these linear partial differential equations we obtain

$$\frac{dr_{12}}{r_{11}} = \frac{dr_{22}}{r_{21}} = \frac{dr_{32}}{r_{31}} = \cdots = \frac{dr_{m2}}{r_{m1}} = \frac{dI}{0},$$

$$\frac{dr_{13}}{r_{12}} = \frac{dr_{23}}{r_{22}} = \frac{dr_{33}}{r_{32}} = \cdots = \frac{dr_{m3}}{r_{m2}} = \frac{dI}{0},$$

$$\frac{dr_{11}}{r_{13}} = \frac{dr_{21}}{r_{23}} = \frac{dr_{31}}{r_{33}} = \cdots = \frac{dr_{m1}}{r_{m3}} = \frac{dI}{0}.$$

Employing the first three fractions in the first row of equalities and writing $\{r_{11}\ r_{22}\}$ for the complementary minor of the minor $|r_{11}r_{22}|$ in the determinant

$$d_{123} = |\ r_{11}r_{22}r_{33}|,$$

* Cf. Amer. Journal of Math., vol. 34, p. 450.

† Gordan: "Ueber Combinanten," Math. Annalen, vol. 5, p. 111.

we have

$$-r_{21}\{r_{11}r_{22}\}dr_{12} + r_{11}\{r_{11}r_{22}\}dr_{22} = 0,$$

$$r_{31}\{r_{11}r_{32}\}dr_{12} - r_{11}\{r_{11}r_{32}\}dr_{32} = 0,$$

$$-r_{31}\{r_{21}r_{32}\}dr_{22} + r_{21}\{r_{21}r_{32}\}dr_{32} = 0.$$

Adding these and denoting by R_{ij} the co-factor of r_{ij} in d_{123},

$$-R_{12}dr_{12} + R_{22}dr_{22} - R_{32}dr_{32} = 0. \tag{7}$$

Likewise from the second and third rows respectively,

$$R_{13}dr_{13} - R_{23}dr_{23} + R_{33}dr_{33} = 0, \tag{8}$$

$$R_{11}dr_{11} - R_{21}dr_{21} + R_{31}dr_{31} = 0. \tag{9}$$

The sum of (7), (8), (9) gives

$$dd_{123} = 0,$$

and hence we have the particular integral

$$d_{123} = C_{123}.$$

Similarly we may obtain C^{m_3} particular integrals

$$d_{ijk} = C_{ijk},$$

and I is a function of the determinants d_{ijk}, as stated.

Of course $I(r)$ is not necessarily rational in the "roots" r_{ij}, but we henceforth assume that I is homogeneous and rational in the determinants d_{ijk}, and either symmetric or alternating in the roots r_{ij}.

The above reasoning may be applied in case of a linearly factorable p-ary form, with similar results.

Following the analogy from the theory of binary invariants expressed in terms of the roots, let us now define a ternary root-difference as the expression obtained by expanding a determinant such as d_{123} on the first column and then dividing each term by the product of the elements of that column. Thus

$$A_{123} \equiv \frac{|r_{11}r_{22}r_{33}|}{r_{11}r_{21}r_{31}} = \frac{|r_{22}r_{33}|}{r_{21}r_{31}} - \frac{|r_{12}r_{33}|}{r_{11}r_{31}} + \frac{|r_{12}r_{23}|}{r_{11}r_{21}}$$

is a ternary root-difference.

We now prove the following theorem which introduces a type of invariant of a factorable ternary form which is also a function of differences:

Any homogeneous function I of the root-differences of a ternary m-line, which is such that in all products of differences of which it consists every root (row of determinant) is involved in the same number of factors, is an invariant of the form.

If every root is involved in the same number of factors in every term of I, the latter becomes upon multiplication by a power of the leading coefficient of the m-line's form a homogeneous function of the determinants d_{ijk}, and as each determinant is an invariant under the general ternary collineation, I will be an invariant. Next suppose that I does not fulfill the above condition, that is, every root does not occur in the same number of factors in the expression for the invariant I. Let us ass me for example $m = 4$ and

$$I = \Sigma A_{123} A_{124},$$

then

$$(r_{11}r_{21}r_{31}r_{41})^2 I = \Sigma r_{31}r_{41}(123)(124),$$

where

$$(123) = |r_{11}r_{22}r_{33}|, \quad \text{etc.}$$

to this apply the linear transformation of determinant -1

$$x_1 = -x_3', \qquad x_2 = x_2', \qquad x_3 = -x_1'.$$

then

$$f_{3m} = \prod_{i=1}^{4} (r_{i1}x_1 + r_{i2}x_2 + r_{i3}x_3),$$

$$f_{3m}' = \prod_{i=1}^{4} (-r_{i3}x_1' + r_{i2}x_2' - r_{i1}x_3'),$$

and we must have of course

$$\Sigma r_{31}r_{41}(123)(124) \equiv \pm \Sigma r_{33}r_{43}(123)(124),$$

which is obviously impossible. Evidently then only those functions are invariant which involve each root in the same number of factors.

Thus in general the form of an m-line invariant, which is also an invariant function of ternary root-differences is

$$I = \Sigma C(r_1r_2r_3)^{a_1}(r_1r_2r_4)^{a_2} \cdots, \tag{10}$$

where each root r_i occurs in the same number of determinant factors in each term of Σ, and when expressed in terms of the coefficients I is annihilated by Δ_{ij}. A similar theory for contravariants is obtained by analogous reasoning.

It is to be noted that no requirement that I should be symmetric in the roots has been imposed. In fact it is often preferable to use an alternating function as an invariant rather than its symmetric square. For when the invariant is expressed in terms of the coefficients of the m-line's form it is often possible to remove an alternating factor, in which case the real invariant has its degree considerably diminished. Thus the invariant

condition that the triangle

$$f_{33} = \prod_{i=1}^{3}(x_1 + r_i x_2 - l_{1-r_i}/l'_{0-r_i}) = a_{0x}{}^3 + a_{1x}{}^2 x_3 + a_{2x} x_3{}^2 + a_{30} x_3{}^3$$

be a pencil is

$$Q = \begin{vmatrix} 1 & r_1 & l_{1-r_1}/l'_{0-r_1} \\ 1 & r_2 & l_{1-r_2}/l'_{0-r_2} \\ 1 & r_3 & l_{1-r_3}/l'_{0-r_3} \end{vmatrix}$$

$$= \sqrt{\Delta}[2a_{01}{}^2 a_{12} - a_{01}a_{02}a_{11} + 9a_{00}a_{03}a_{11} - 6a_{01}a_{03}a_{10} + 2a_{02}{}^2 a_{10} - 6a_{00}a_{02}a_{12}] = 0,$$

where Δ is the discriminant of the binary cubic $a_{0x}{}^3$. This is the most elementary full invariant of a plane m-line and is only of the third degree in the a's. Since it is of the first degree in the coefficients of $a_{1x}{}^2$, it follows that in the space of nine dimensions whose coördinates are all the coefficients of ternary forms of order three, the forms representing pencils fill a rational five dimensional spread included in the known rational spread* of six dimensions representing those forms which are linearly factorable.

When in a p-ary form $m = p$, the one full invariant of the m-line always has the form of a single determinant, and therefore the simplest full invariant of a binary quadratic is the condition for a double point; of a ternary 3-line, the condition for a triple point; of a quaternary quartic, the condition for a quadruple point, etc.

A 2-line has no full invariant, but it has a contravariant, representing its double point. Let

$$f_{32} = \prod_{i=1}^{2}(x_1 + r_i x_2 - l_{1-r_i}/l'_{0-r_i}) = 0$$

be the 2-line. Then its contravariant, readily computed by use of (1) and symmetric functions, is

$$U_2 = \sqrt{4a_{00}a_{02} - a_{01}{}^2}[(2a_{02}a_{10} - a_{01}a_{11})u_1 + (2a_{00}a_{11} - a_{01}a_{10})u_2 + (4a_{00}a_{02} - a_{01}{}^2)u_3] = 0.$$

We may readily derive the conditions that two 2-lines f_{32}, g_{32} (§ 1) form a harmonic pencil. In fact, if we abbreviate U_2, and the corresponding contravariant of g_{32} as

$$U_2 = \sqrt{D}[\lambda_1(a)u_1 + \lambda_2(a)u_2 + \lambda_3(a)u_3] = 0,$$
$$V_2 = \sqrt{D'}[\lambda(b)u_1 + \lambda_2(b)u_2 + \lambda_3(b)u_3] = 0,$$

and let C_1 be the seminvariant C_1 from set (2) § 1, the proposed conditions are the following three:

(11) $\lambda_1(a)\lambda_2(b) - \lambda_1(b)\lambda_2(a) = 0, \quad \lambda_1(a)\lambda_3(b) - \lambda_1(b)\lambda_3(a) = 0, \quad C_1 = 0.$

* Transactions Amer. Math. Soc., vol. 12, p. 368.

3. Special Complete Systems.—An invariant of

$$f_{3m} = \prod_{i=1}^{m} (r_{i1}x_1 + r_{i2}x_2 + r_{13}x_3)$$

has been shown to be a function of third order determinants of the general form

$$R = \Sigma(r_1r_2r_3)^{\alpha_1}(r_1r_2r_4)^{\alpha_2} \cdots (r_{m-2}r_{m-1}r_m)^{\alpha_m c_3},$$

where R is assumed to be integral and rational. We have proved concerning any homogeneous invariant function of ternary differences, I, that in every term of I each r_k occurs in exactly the same number of determinant factors as every other r. Under these defining conditions the number of distinct types of I will be finite. For they lead to a set of diophantine equations to be satisfied in positive integers:

$$\alpha_i + \alpha_j + \cdots = \alpha_{i1} + \alpha_{j1} + \cdots,$$
$$\cdots\cdots\cdots\cdots$$

and the totality of solutions of such a system is expressible in terms of a finite number.*

We proceed to special cases.

(i) $m = 3.$

Here the general form of R is

$$R_3 = (r_1r_2r_3)^{\alpha_1}$$

Hence every invariant of f_{33} is a power of

$$I_1 = (r_1r_2r_3)$$

which has been computed in terms of the coefficients. (See Q, § 2.)

The contravariants are

$$C_3 = \Sigma(r_1r_2u)^{\beta_1}(r_2r_3u)^{\beta_2}(r_3r_1u)^{\beta_3},$$

and from the defining conditions we have

$$\beta_1 + \beta_2 = \beta_1 + \beta_3 = \beta_2 + \beta_3.$$

Hence $\beta_1 = \beta_2 = \beta_3$ and the only contravariant is

$$I_2 = (r_1r_2u)(r_2r_3u)(r_3r_1u).$$

To interpret I_2 consider it as the product of three eliminants of pairs of factors of f_{33} taken with

$$u_x \equiv u_1x_1 + u_2x_2 + u_3x_3,$$

which is evidently the line equation of the vertices of f_{33}.

* The reader should consult Hilbert's paper on the binary case, Math. Annalen, vol. 33; also Cayley, Math. Annalen, vol. 34.

(ii) $$m = 4.$$

The general form of R for this case is

$$R_4 = \Sigma(r_1r_2r_3)^{\alpha_1}(r_1r_2r_4)^{\alpha_2}(r_1r_3r_4)^{\alpha_3}(r_2r_3r_4)^{\alpha_4}.$$

Hence $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$. Therefore every invariant of f_{34} is a power of

$$I_1 = (123)(124)(134)(234).$$

The contravariants are

$$C_4 = \Sigma(r_1r_2u)^{\beta_1}(r_1r_3u)^{\beta_2}(r_1r_4u)^{\beta_3}(r_2r_3u)^{\beta_4}(r_2r_4u)^{\beta_5}(r_3r_4u)^{\beta_6}.$$

Hence $\beta_4 = \beta_3$, $\beta_5 = \beta_2$, $\beta_6 = \beta_1$.
Therefore there are three contravariants

$$I_2 = \Sigma(r_1r_2u)(r_3r_4u),$$

$$I_3 = \Sigma(r_1r_2u)(r_1r_3u)(r_2r_4u)(r_3r_4u),$$

$$I_4 = (r_1r_2u)(r_1r_3u)(r_1r_4u)(r_2r_3u)(r_2r_4u)(r_3r_4u).$$

(iii) $$m = 5.$$

Here

$$R = \Sigma(123)^{\alpha_1}(124)^{\alpha_2}(125)^{\alpha_3}(134)^{\alpha_4}(135)^{\alpha_5}(145)^{\alpha_6}(234)^{\alpha_7}(235)^{\alpha_8}(245)^{\alpha_9}(345)^{\alpha_{10}}.$$

and we are able to express all the α's in terms of six independent ones. In fact

$$R = \Sigma(123)^{1/2(2\alpha_6-\alpha_7-\alpha_8+\alpha_9+\alpha_{10})}(124)^{1/2(2\alpha_5-\alpha_7+\alpha_8-\alpha_9+\alpha_{10})}(125)^{1/2(-2\alpha_5-2\alpha_6+3\alpha_7+\alpha_8+\alpha_9+\alpha_{10})}$$
$$(134)^{-\alpha_5-\alpha_6+\alpha_7+\alpha_8+\alpha_9}(135)^{\alpha_5}(145)^{\alpha_6}(234)^{\alpha_7}(235)^{\alpha_8}(245)^{\alpha_9}(345)^{\alpha_{10}}.$$

Taking $\alpha_i = 0, 1, 2$ we obtain the following invariants:

$$I_1 = \Sigma(123)(125)(134)(245)(345),$$
$$I_2 = \Sigma(125)(134)^2(235)(245),$$
$$I_3 = \Sigma(125)^3(134)^3(234)(235)(245)(345),$$
$$I_4 = \Sigma(123)(125)^2(134)^2(145)(234)(235)(245)(345),$$
$$I_5 = (123)(124)(125)(134)(135)(145)(234)(235)(245)(345),$$
$$I_6 = \Sigma(123)(124)(125)^3(134)^4(235)^2(245)^2(345)^2,$$
$$I_7 = \Sigma(123)^3(124)(125)(134)^2(145)^2(235)^2(245)^2(345)^2,$$
$$I_8 = \Sigma(123)(124)(125)(134)^2(135)^2(145)^2(234)^2(235)^2(245)^2,$$
$$I_9 = \Sigma(123)^3(125)(134)(145)(245)^2(345)^2,$$
$$I_{10} = \Sigma(123)^2(124)(125)(134)(135)(245)^2(345)^2.$$

Through the medium of (1) § 1 all of these invariants and covariants have expressions in terms of the actual coefficients.

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PLANE CURVES WITH CONSECUTIVE DOUBLE POINTS.

By F. R. Sharpe and C. F. Craig.

The problem of the existence of plane curves of given order and genus has been completely solved for the case in which all the singularities are ordinary double points and are distinct.* Guccia has raised the question† as to how many of these double points can be consecutive. The problem has already been solved for the quartic and the quintic.‡ In this paper a method is developed for determining a curve of order n with d consecutive double points. A lower limit for the maximum value of d is determined. The equation of the rational sextic with all its double points consecutive is given.

I. The General Problem.

1. In this section a method for determining the equation of a plane curve of order n having d consecutive double points will be deduced. Let the equation of the curve C_n be taken in the form

$$\sum_{k=0}^{n} \sum_{i=0}^{n-k} A_{ki} x^k y^i = 0, \tag{1}$$

where the coefficients A_{ki} are to be determined. Let the point $(0, a_0)$ be the double point and let the equation of one of the branches through this point be

$$y = \sum_{t=0}^{\infty} a_t x^t. \tag{2}$$

If this value of y is substituted in (1), then

$$A_p = 0, \quad p = 0, 1, 2, \cdots, \tag{3}$$

where A_p denotes the coefficients of x^p in the resulting identity in x.

2. To obtain the explicit form of A_p, write

$$y^i = \sum_{p=k}^{\infty} B_{i,\, p-k} x^{p-k},$$

where

$$B_{i,\, p-k} = (i!) \Sigma \frac{\Pi a_t^{i_t}}{\Pi(i_t!)}, \tag{4}$$

* See V. Snyder, "Construction of plane curves of given order and genus, having distinct double points," Bulletin Amer. Math. Soc., Series 2, vol. 15, 1908, pp. 1–4.

† See Circ. Mat. Palermo, Supplemento ai Rendiconti, vol. 5, 1910, p. 1.

‡ See Salmon, Higher Plane Curves.

in which the two products are taken with respect to i_t, $(i_t!) = 1$ if $i_t = 0$, and the summation extends over all solutions in positive integers and zero of

$$(5) \qquad \begin{aligned} &i_0 + i_1 + i_2 + \cdots + i_t + \cdots + i_{p-k} = i, \\ &i_1 + 2i_2 \cdots + ti_t + \cdots + (p-k)i_{p-k} = p - k. \end{aligned}$$

When this value of y^i is substituted in (1), the equations (3) become

$$(6_p) \qquad A_p = \sum_{k=0}^{n} \sum_{i=0}^{n-k} A_{ki} B_{i,\, p-k} = 0, \; p = 0, \; 1, \; 2, \; \cdots.$$

In these equations $B_{i,q} = 0$ if $q < 0$. By assigning to p the successive values 0, 1, 2, $\cdots$, sufficient linear equations in the unknown coefficients A_{ki} will be obtained to completely determine them. The rest of the equations (6_p) will then determine the remaining coefficients a_t of (2), beginning with a certain one of them, in terms of the others.

3. To determine the additional equations of condition on the A_{ki}, in order that C_n shall have double points, requires a discussion of the equations (6_p). From (5) it follows that *a_t can enter only linearly in A_p if $2t > p$ unless $t > p$, in which case a_t does not enter into the value of A_p.*

4. If in (6_p), we take $2s > p > s$, then the part of the coefficient of a_{p-s} in A_p which contains no a_t with $t > s$ is

$$(7) \qquad a_s \sum_{i=0}^{n} \frac{i!}{(i-2)!} a^{i-2} A_{0i} + T_s,$$

where

$$T_s \equiv \sum_{k=0}^{n} \sum_{i=0}^{n-k} A_{ki} B'_{i,\, s-k},$$

in which $B'_{i,\, s-k}$ has the same form as (4) except that the summation is taken so that

$$i_0 + i_1 + i_2 + \cdots + i_{s-1} = i - 1,$$

$$i_1 + 2i_2 + \cdots + (s-1)i_{s-1} = s - k.$$

If $p = 2s$, if follows from (4), (5) and (6) that the part of A_{2s} which contains no a_t which $t > s$ is

$$(8) \qquad A_{2s}' \equiv a_s^2 \sum_{i=0}^{n} \frac{i!}{2!(i-2)!} a_0^{i-1} A_{0i} + a_s T_s + T_s',$$

where T_s has the same significance as in (7) and

$$T_s' \equiv \sum_{k=0}^{n} \sum_{i=0}^{n-k} A_{ki} B''_{i,\, 2s-k},$$

in which the summation for $B''_{i,\,2s-k}$ is taken so that

$$i_0 + i_1 + i_2 + \cdots + i_{s-1} = i,$$
$$i_1 + 2i_2 + \cdots + (s-1)i_{s-1} = 2s - k.$$

If $s = 0$, (7) and (8) become respectively

$$(7') \qquad \sum_{i=0}^{n} i a_0^{i-1} A_{0i},$$

$$(8') \qquad A_0 \equiv \sum_{i=0}^{n} a^i A_{0i}.$$

5. The successive conditions that C_n shall have 1, 2, $\cdots$, d double points can now be written down. In doing this it is desirable, for the moment, to think of the A_{ki} in the equations (6_p) as being known and of the coefficients a_t as unknown. Since $(0, a_0)$ is a double point on C_n, it counts for two of the intersections of (1) and $x = 0$; that is $A_0 = 0$ must have a_0 as a double root. Hence

$$(9_0) \qquad \sum_{i=0}^{n} a_0^i A_{0i} = 0$$

and

$$(9') \qquad \sum_{i=0}^{n} i a_0^{i-1} A_{0i} = 0$$

hold simultaneously. From § 3, $(9_0')$ and (7') it follows that *if C_n has one double point at $(0, a_0)$ A_p contains no a_t with $t > p - 1$.*

6. From the hypothesis that C_n has d consecutive double points at $(0, a_0)$ it follows that there is a second branch of the curve through this point, and that the equation of this branch is

$$(2') \qquad y' = \sum_{t=0}^{d-1} a_t x^t + \sum_{t=d}^{\infty} a_t' x^t.$$

The equations which determine the coefficients $a_1, a_2, \cdots, a_{d-1}$ must therefore have these coefficients as double roots. These equations are quadratic in the unknown and the following lemma will be used. *If $ax^2 + 2bx + c = 0$ has double roots, $ax + b = 0$ and $bx + c = 0$ are consistent, and conversely.*

7. From § 5 it follows that (6_1) no longer contains a_1. If $s = 1$, by using (8) and § 5 we reduce (6_2) to $A_2' = 0$. In order that C_n may have two consecutive double points at $(0, a_0)$, $A_2' = 0$ must have equal roots. Using the lemma of § 6, $A_2' = 0$ and the condition for equal roots are replaced by the equations

$$(9_2) \qquad a_1 \sum_{i=0}^{n} i(i-1) a_0^{i-1} A_{0i} + T_1 = 0,$$

$$(9_2') \qquad a_1 T_1 + 2T_1' = 0.$$

If $s = 1$, by combining (9_2) and (7) we deduce that A_{p-1}, $p > 2$, no longer contains a_{p-1}. Hence from § 5, *if C_n has two consecutive double points, A_p contains no a_t with $t > p - 2 > 0$.*

8. It should here be noticed that the equations (9_0), $(9_0')$ and (6_1) must be satisfied in order that C_n may have one double point, and that the three equations (9_2), $(9_2')$ and (6_3) must also be satisfied to ensure two consecutive double points. Each additional consecutive double point requires three conditions.

9. The argument of § 7 may be repeated with $s = 2$. The conclusion is that if C_n has three consecutive double points, A_p contains no a_t with $t > p - 3 > 0$. If the argument is repeated a sufficient number of times, the equations of condition, in order that C_n take on the last of $s + 1$ consecutive double points, are

$$(9_{2s}) \qquad a_s \sum_{i=0}^{n} i(i-1)a_0^{i-1}A_{0i} + T_s = 0,$$

$$(9_{2s}') \qquad a_s T_s + 2T_s' = 0$$

and (6_{2s+1}). From (7) it follows that *if C_n has $s + 1$ consecutive double points, A_p will contain no a_t with $t > p - s + 1 > 0$.*

10. The curve (1) will have d consecutive double points if the conditions given by equations (9_{2s}), $(9_{2s}')$ and (6_{2s+1}) with $s = 0, 1, 2, \cdots, d - 1$ are satisfied. None of the equations of condition will contain a coefficient a_t with $t > s$. Each time a new double point is added, the equations of condition will contain the a_s corresponding to this double point linearly. These $3d$ equations of condition will be referred to collectively as the equations (E) and are equations for determining the $\frac{1}{2}n(n + 3)$ unknown coefficients A_{ki} of (1).

11. If $D \equiv 3d - \frac{1}{2}n(n + 3) > 0$, there are D more equations than there are non-homogeneous unknowns A_{ki}. The least value of n for which this inequality can hold is $n = 6$, with $d = 10$. There is no problem if $n < 6$ and hence there are at least *seven* consecutive double points if $n \geq 6$. It is therefore no restriction to take the expansion (2) in the canonical form*

$$(10) \qquad y = \tfrac{1}{2}x^2 + x^5 + \sum_{t=7}^{\infty} a_t x^t.$$

The necessary choice of the coördinate system uses all the projective constants in the expansion (2). The equations (E) are now supposed to be specialized to conform to the expansion (10).

12. Since there is no higher singularity at the origin (a_0 is now zero) than a double point it is no restriction to assume $A_{01} = 1$. If $D > 0$,

* See E. J. Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces, p. 84.

there are more equations than unknowns A_{ki} in the system (E). For these equations to be consistent requires the vanishing of D linearly independent determinants of order $\frac{1}{2}(n+1)(n+2)$ of the coefficients of A_{ki}. These determinants are rational integral functions of the $d-7$ coefficients a_t, $t = 7, 8, \cdots, d-1$. Hence the number of these equations can not exceed the number of the coefficients a_t. Thus

$$D \leq d - 7$$

and therefore

$$d \leq \tfrac{1}{4}(n^2 + 3n - 14). \tag{11}$$

This inequality gives a maximum for the value of d unless the curve C_n thus determined has additional consecutive double points.

13. The D equations of § 12, connecting the coefficients a_t of (10) have a number of simultaneous solutions. In case a solution of these equations is such that (10) is the expansion of a curve of order less than $n/2$, this solution must be rejected since the curve (1) is then composite. If the expansion (10) which corresponds to a solution of these equations is that of a curve of order greater than $n/2$, the curve (1) can not be composite. When n is even, the equation (1) may represent a curve of order $n/2$ counted twice of which (10) is the expansion given by a solution of these D equations.

II. The Sextic Curves.

14. The upper limit for d given by (11) is too low to assure the existence of rational curves of order higher than six with all their double points consecutive. The curves of order six with ten consecutive double points will now be determined.

15. Instead of starting with the sextic in the form (1) a known* result will be specialized. If $c_3 = 0$, and $c_3' = 0$, define a pencil of cubic curves with eight consecutive basis points and if $c = 0$ is a sextic, which is not a quadratic function of the cubics, having these eight basis points for double points then

$$Ac_3^2 + Bc_3c_3' + C(c_3')^2 + Dc = 0$$

will represent ∞^3 sextics with eight consecutive double points, the coefficients being constants. The curve c will satisfy the necessary conditions if it is the product of a conic $c_2 = 0$ through the first five basis points of the pencil of cubics and a quartic $c_4 = 0$ which has the remaining three consecutive basis points of the pencil for double points and one of its branches intersecting the pencil in the first five consecutive basis points.

* See V. Snyder, "The involutorial birational transformation of the plane, of order 17," Amer. Jour. Math., vol. 33, 1908, pp. 327–336; see p. 328.

16. Let the expansion (10) be written in the form

$$y = \tfrac{1}{2}x^2 + x^5 + Tx^7 + Ux^8 + Vx^9 + Wx^{10} + \cdots. \tag{12}$$

The pencil of cubics is defined by*

$$c_3 \equiv y - \tfrac{1}{2}x^2 - 2Ty(y - \tfrac{1}{2}x^2) - 4xy^2 = 0,$$

$$c_3' \equiv x(y - \tfrac{1}{2}x^2) - 8y^3 = 0;$$

the conic is

$$c_2 \equiv y - \tfrac{1}{2}x^2 = 0;$$

the quartic is

$$c_4 \equiv T(y - \tfrac{1}{2}x^2)^2 - 2xy(y - \tfrac{1}{2}x^2) + 16y^4 = 0;$$

and the sextics are

$$Ac_3^2 + Bc_3c_3' + C(c_3')^2 + Dc_2c_4 = 0. \tag{13}$$

17. If the equations (E) of § 10 are formed for this sextic, it will be found that all are satisfied except those for which $s = 8$ and $s = 9$ respectively. These are

$$2A(U - 4) + BT - D = 0; \tag{14}$$

$$BT(U - 4) + 2CT^2 - D(U - 8) = 0; \tag{15}$$

$$B(U - 4)(U - 6) + 2CT(U - 6) + DT^2 = 0; \tag{16}$$

$$A(V - T^2) + B(U - 5) + CT = 0; \tag{17}$$

$$B[(V - T^2)(U - 5) + T^2(U - 4)] + C[T(V - T^2) + (U - 6)^2 + 2T^3] + 4DT = 0; \tag{18}$$

$$B[V(V - T^2) + TU(U - 4)] + C[2T^2U + 2V(U - 6)] + D[20 + 3T^3 + U(6 - U)] = 0. \tag{19}$$

18. From (15) and (16) is obtained, since $D \neq 0$,

$$T^3 + (U - 6)(U - 8) = 0; \tag{20}$$

from (14), (15) and (17) follows

$$A[T(V - T^2) - (U - 4)(U - 6)] + D(U - 7) = 0; \tag{21}$$

from (14), (15) and (18) comes

$$A[(U - 4)^2(U - 6)^2 - T(V - T^2)(U - 4)(U - 6)] + D[VT(U - 7) + 3T^3 - 2(U - 6)^2] = 0; \tag{22}$$

* See Wilczynski, loc. cit., p. 67. All four curves may be constructed by the method of part I, or by the method given by Salmon.

and (19) becomes

$$\begin{aligned}(23)\qquad & A[2V(U-4)^2(U-6)-2VT(V-T^2)(U-4)]\\ & +D[VT(V-T^2)-4V(U-6)-2T^2U+20T^2+3T^5]=0.\end{aligned}$$

From (20), (21) and (22) is obtained

$$(24)\qquad VT=-\frac{(U-6)(U-8)^2}{U-7},$$

and (23) gives

$$(25)\qquad T^2[U(T^3+2)-(T^3+3)(T^3+4)]=0.$$

The solution $T=0$ is rejected since (15) would violate $D\neq 0$. The second factor of (25) combined with (20) gives

$$(26)\qquad T^6+T^3-1=0.$$

In succession follow

$$(27)\qquad \begin{aligned}U&=7-T^3,\\ V&=2T^2+T^5,\\ B&=-A(5T^2+4T^5),\\ C&=A(5T+6T^4),\\ D&=A(2-3T^3).\end{aligned}$$

The equation $A_{20}=0$ becomes, after the usual reduction,

$$\begin{aligned}&A[W-6T-TU]^2+D[6UT^2-2UV+6V-2T^2]\\ &\quad+B[(2V-T^2)(W-6T-TU)-2(U^2-16)+VT(U-4)+UT(V-T^2)]\\ &\qquad+C[2(U-6)(W-6T-TU)+V^2-16TU+2TU^2+2T^2V-16T]=0,\end{aligned}$$

from which

$$(28)\qquad W=\frac{42T-7T^4\pm\sqrt{-39T^2+63T^5}}{2}.$$

19. *If T satisfies* (26) *the curves*

$$c_3^2-(5T^2+4T^5)c_3c_3'+(5T+6T^4)(c_3')^2+(2-3T^3)c_2c_4=0$$

are rational sextics having ten consecutive double points. The two branches of each of these curves have the expansion (12) *where T, U, V, W have the values given by* (26), (27) *and* (28). *All other rational sextics having ten consecutive double points are projectively equivalent to these.*

20. From (14) and (15) the values of B and C may be substituted in (13) and there results *the curves*

$$A[Tc_3-(U-4)c_3']^2+D[Tc_3c_3'-2(c_3')^2+T^2c_2c_4)=0,$$

where T and U satisfy (20), *have nine consecutive double points. If $A=0$ there is a triple point at* (0, 0).

CORNELL UNIVERSITY,
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THE EFFECT OF RADIATION ON A SMALL PARTICLE REVOLVING ABOUT JUPITER.

By Theodore Henry Brown.

1. Since the experiments of Nichols and Hull* on radiation pressure there has arisen the question of the effect of radiation in problems of celestial mechanics. Some of these problems have been already solved. Professor J. H. Poynting† has worked out the conditions of size and temperature for two bodies in equilibrium in space and also solved to a first approximation the problem of two bodies moving under the pressure of radiation and the Newtonian attraction. Professor E. B. Wilson‡ has solved the equations for this problem to a fourth approximation.

The present paper treats the motion of three bodies in one plane acting under radiation pressure as well as ordinary Newtonian forces: the third body being taken as a small satellite of Jupiter. The method of solution differs from the previous ones in that the time is not eliminated from the equations. The general effect of radiation on the particle is similar to that obtained in the above papers, namely that the particle is drawn in toward the attracting center losing eccentricity and reaching the surface after a finite number of revolutions. There are, however, some new effects. Considering the Sun to be stationary, it is clear that the particle will be in the shadow of Jupiter during a part of each revolution. It is found that the effect of the eclipse on the motion is to draw the particle in toward Jupiter and to diminish the eccentricity. But now suppose the Sun is moving. Then, if the motion of the particle is direct, the amount of shadow that the particle has to pass through will be greater than before. The particle will still be drawn in and lose eccentricity but the amount is not so great. On the other hand, if the motion of the particle is retrograde, the shadow will be less than before. Consequently the amount the particle is drawn in and the amount the eccentricity loses is greater. This shows that the number of revolutions which a particle can make with a retrograde motion is less than that which a similar particle can make with a direct motion.

2. The assumptions in regard to the radiation pressure will be the same

* "The pressure due to radiation," Astrophysical Journal, vol. 17, June, 1903.

† Nature, vol. 75, pp. 90–93. Also "Radiation in the solar system: its effect on temperature and its pressure on small bodies," Trans. Royal Society, ser. A, vol. 202 (1903).

‡ "The revolution of a dark particle about a luminous center," Annals of Math., 2d ser. vol. 7–8 (1905–07).

as in Professor Wilson's paper. That is in addition to the usual gravitational forces there is the direct pressure of light and the Doppler reception and emission effects. Besides these we shall assume the motion is all in one plane and that the particle is started well within the sphere of action of Jupiter which is taken to be one twentieth the distance from Jupiter to the Sun.

Let Jupiter be the origin of coördinates. Denote it by B and its mass by m_2. Let S be the Sun of mass m_1 and P the particle of mass m_3. Then let $BP = r$, $BS = R$, $SP = l$, $\angle PBS = \varphi$ and after Jupiter is reduced to rest let Bx be a line fixed in the plane of motion so that $\angle PBx = \vartheta$, $\angle SBx = \alpha$. Then, if V be the velocity of light, K the constant of radiation referred to S and K' that referred to B, we have for the equations of motion of the particle:*

$$\frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2 = -\frac{m_2}{r^2} - \frac{1}{3}\frac{K}{l^2}\frac{1}{V}\frac{dr}{dt} + \frac{m_1}{l^3}R\left(\cos\varphi - \frac{r}{R}\right) - \frac{m_1}{R^2}\cos\varphi$$
$$- \frac{K}{l^3}R\left(\cos\varphi - \frac{r}{R}\right) + \frac{K}{l^3}\frac{1}{V}\frac{dl}{dt}R\left(\cos\varphi - \frac{r}{R}\right)$$
$$+ \frac{K}{r^2} - \frac{4}{3}\frac{K'}{V}\frac{1}{r^2}\frac{dr}{dt}.$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right) = -\frac{1}{3}\frac{K}{l^2}\frac{1}{V}r\frac{d\vartheta}{dt} - \frac{m_1}{l^3}R\sin\varphi + \frac{K}{l^3}R\sin\varphi$$
$$- \frac{K}{l^3}\frac{1}{V}\frac{dl}{dt}R\sin\varphi + \frac{m_1}{R^2}\sin\varphi - \frac{1}{3}\frac{K'}{V}\frac{1}{r^2}\frac{rd\vartheta}{dt}.$$

Since it is only possible to solve these equations by series and since we desire an approximation involving only the first power of the disturbance from elliptic motion, we shall divide the equations into two parts, integrate each and add the results. The division is such that the first equations, which have a potential, represent the elliptic motion of P disturbed by the attraction of the Sun and the direct pressure of light from the Sun while the second set, which has no potential, gives the motion disturbed by the Doppler effects and the direct pressure from Jupiter.

Take as the first set of equations

$$\frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2 = -\frac{m_2}{r^2} + \left[\frac{m_1}{l^3}R\left(\cos\varphi - \frac{r}{R}\right)\right.$$
$$\left. - \frac{m_1}{R^2}\cos\varphi - \frac{K}{l^3}R\left(\cos\varphi - \frac{r}{R}\right)\right]$$
$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right) = \left[\frac{m_1}{R^2}\sin\varphi - \frac{m_1}{l^3}R\sin\varphi + \frac{K}{l^3}R\sin\varphi\right].$$

* See E. B. Wilson, Annals of Math., 2d ser., vol. 7–8 (1905–07), pp. 135–137.

The right-hand sides may be expressed as partial derivatives of the function

$$F = \frac{m_2}{r} + \frac{m_1}{l} - \frac{K}{l} - \frac{m_1}{R^2} r \cos \varphi$$

$$= \frac{m_2}{r} - \frac{K}{R}\left(\frac{r}{R}\right)\cos \varphi + \frac{m_1 - K}{R}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 (3 \cos^2 \varphi - 1) + \cdots\right]$$

$$= \frac{m_2}{r} + \mathcal{R}.$$

To solve the equations we shall develop $\mathcal{R}$* in terms of the elliptic elements and then substitute it in the equations for the variation of the elliptic elements. These equations,† when the powers of e above the second are neglected, become

$$\frac{da}{dt} = \frac{2na^2}{m_2}\frac{\partial \mathcal{R}}{\partial \epsilon},$$

$$\frac{de}{dt} = -\frac{nae}{2m_2}\frac{\partial \mathcal{R}}{\partial \epsilon} - \frac{na(2 - e^2)}{2m_2 e}\frac{\partial \mathcal{R}}{\partial \varpi},$$

$$\frac{d\varpi}{dt} = \frac{na(2 - e^2)}{2m_2 e}\frac{\partial \mathcal{R}}{\partial e},$$

$$\frac{d\epsilon_1}{dt} = -\frac{2na^2}{m_2}\frac{\partial \mathcal{R}}{\partial a} + \frac{nea}{2m_2}\frac{\partial \mathcal{R}}{\partial e}.$$

After substituting and integrating with respect to the time treating the elements on the right-hand sides as constants we get the first approximation to the changes in the elliptic elements due to the disturbing forces. These are, writing the first few terms:

$$\delta a = (m_1 - K)\frac{2na^4}{m_2 a'^3}\left\{\left[\frac{3}{2} - \frac{15}{4}e^2\right]\frac{\cos(2l + 2g - 2l')}{2(n - n')} - \frac{e}{2n}\cos l + \cdots\right\}$$

$$+ K\frac{2na^3}{m_2 a'^2}\left\{\left[\frac{1}{2}e^2 - 1\right]\frac{\cos(l' - l - g)}{n - n'} + \frac{e}{n' - 2n}\cos(l' - 2l - g) + \cdots\right\}$$

$$= (m_1 - K)\frac{2na^4}{m_2 a'^3}\{E_1'\} + K\frac{2na^3}{m_3 a'^2}\{E_1''\}$$

say,

* Delaunay, Mém. de l'Acad. des Sci., vol. XXVIII (1860), pp. 33–54.

† Tisserand, Mécanique Céleste, vol. I, p. 169.

$$\delta e = -(m_1 - K)\frac{nea^3}{2m_2a'^3}\{E_1'\} - K\frac{nea^2}{2m_2a'^2}\{E_1''\}$$

$$-(m_1-K)\frac{(2-e^2)na^3}{2m_2ea'^3}\left\{\frac{e}{2n}\cos l - \frac{3e}{4(3n-2n')}\cos(3l+2g-2l')+\cdots\right\}$$

$$-K\frac{(2-e^2)na^2}{2m_2ea'^2}\left\{-\frac{3e}{2n'}\cos(l-g) - \frac{e}{2(n'-2n)}\cos(l-2l-g)+\cdots\right\}$$

$$e\delta\varpi = (m_1-K)\frac{(2-e^2)na^3}{2m_2a'^3}\left\{\frac{3}{4(3n-2n')}\sin(3l+2g-2l') - \frac{1}{2n}\sin l\right.$$

$$\left. - \frac{9}{4(n-2n')}\sin(l-2l') + \frac{3}{4}et + \cdots\right\}$$

$$+K\frac{(2-e^2)na^2}{2m_2a'^2}\left\{\frac{3}{2n'}\sin(l'-g) - \frac{1}{2(n'-2n)}\sin(l'-2l-g)+\cdots\right\}$$

$$= (m_1-K)\frac{(2-e^2)na^3}{2m_2a'^3}\{E_2'\} + K\frac{(2-e^2)na^2}{2m_2a'^2}\{E_2''\}$$

say,

$$\delta\epsilon_1 = -(m_1-K)\frac{4na^3}{m_2a'^3}\left\{\frac{1}{4}t + \left[\frac{3}{4}-\frac{15}{8}e^2\right]\frac{\sin(2l+2g-2l')}{2(n-n')}+\cdots\right\}$$

$$-K\frac{2na^2}{m_2a'^2}\left\{\left[\frac{1}{2}e^2-1\right]\frac{\sin(l'-l-g)}{n'-n}+\frac{3e}{2n'}\sin(l'-g)+\cdots\right\}$$

$$+(m_1-K)\frac{nea^3}{2m_2a'^3}\{E_2'\} + K\frac{nea^2}{2m_2a'^2}\{E_2''\},$$

where l is the mean anomaly and g the longitude of perigee, the prime letters referring to the Sun.

We take as the second set of equations:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2 = -\frac{m_2}{r^2} - \frac{1}{3}\frac{K}{l^2}\frac{1}{V}\frac{dr}{dt} + \frac{K}{l^3}\frac{1}{V}\frac{dl}{dt}R\left(\cos\varphi - \frac{r}{R}\right) + \frac{K'}{r^2} - \frac{4}{3}\frac{K'}{V}\frac{1}{r^2}\frac{dr}{dt}$$

$$= -\frac{m_2}{r^2} + P$$

say,

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right) = -\frac{1}{3}\frac{K}{l^2}\frac{1}{V}r\frac{d\vartheta}{dt} - \frac{K}{l^3}\frac{1}{V}\frac{dl}{dt}R\sin\varphi - \frac{1}{3}\frac{K'}{V}\frac{1}{r^2}\frac{rd\vartheta}{dt}$$

$$= T$$

say.

The solution may be obtained by expanding the right-hand sides in terms of the elliptic elements and substituting in the equations for the

variation of elliptic elements which contain the forces on the right-hand sides. The expansion of dl/dt and $1/l^3$ in terms of r/R and $\cos k\varphi$ is effected as usual in the development of the perturbative function, the latter in turn being expanded in terms of the elements by use of the elliptic developments. The expansions are, to the order of e desired,

$$P = g_1\left\{\frac{na}{2}[1 + e\cos l]\sin 2\varphi - \frac{n'a}{2}[1 - e\cos l]\sin 2\varphi - na[e\sin l]\cos^2\varphi\right\} - g_1\frac{na}{3}[e\sin l] + \frac{k}{a^2}\left[1 + 2e\cos l + \frac{1}{2}e^2 + \frac{5}{2}e^2\cos 2l\right] - 4g_2\frac{ne}{a}\sin l,$$

$$T = g_1\left\{\left[-na(1 + e\cos l) + n'a(1 - e\cos l) + \frac{na}{2}e\sin l\right]\left[\frac{1}{2} - \frac{1}{2}\cos 2\varphi\right]\right\} - g_1\frac{na}{3}[1 + e\cos l] - g_2\frac{n}{a}[1 - 3e\cos l],$$

where $g_1 = K/R^2V$, $g_2 = K'/3V$ and where φ has yet to be expanded in terms of the elements by use of the expression

$$\varphi = \vartheta - (n't + \epsilon').$$

Neglecting higher powers the equations to be used become

$$\frac{da}{dt} = \frac{2na^2}{m_2}\left[Pae\sin\vartheta + T\frac{a^2}{r}\right],$$

$$\frac{de}{dt} = \frac{na}{m_2}\left[Pa\sin\vartheta + T\left(\frac{a^2}{r} - r - \frac{3}{2}\frac{a^2}{r}e^2 + \frac{1}{2}re^2\right)\frac{1}{e}\right],$$

$$e\frac{d\varpi}{dt} = \frac{na}{m_2}[-Pa\cos\vartheta + T(a + r)\sin\vartheta],$$

$$\frac{d\epsilon_1}{dt} = -\frac{2nar}{m_2}P.$$

In making the substitutions it will be found easier to substitute the expressions for P and T before expanding the functions of φ and ϑ into series of the elements. After integrating as before we get the first approximation to the changes due to these disturbing forces. These are, writing only the first few terms:

$$\delta a = ag_1\left[\frac{n'}{n} - 1\right]t + \frac{ag_1}{2n}\sin 2(l - l') - \frac{2ag_1e}{n}\sin l - \frac{2}{3}g_1at - g_1\frac{4ae}{3n}\sin l - \frac{2naK'e}{m_2n}[\cos l - e\cos 2l] - g_2\frac{2t}{a} - g_2\frac{8e}{an}\sin l + \cdots,$$

$$\delta e = -\frac{5}{4}\frac{n'}{n} g_1 e t + \frac{g_1}{4}\left[e + 2 - 2\frac{n'}{n}\right]\frac{\sin(3l - 2l')}{6n - 4n'} + \frac{g_1}{n}\left[\frac{n'}{n} - 1\right]\sin l$$
$$+ \frac{g_1}{4}\left[e + 3 - 3\frac{n'}{n}\right]\frac{\sin(2l' - l)}{4n' - 2n} + \frac{g_1 e}{4}\left[\frac{n'}{n} + 2\right]\frac{\sin 2(l - l')}{2(n - n')}$$
$$- \frac{2g_1}{3n}\sin l - \frac{g_1 e}{3n}\sin 2e - \frac{K'}{m_2}\left[\left(1 + \frac{7}{8}e^2\right)\cos l + e\cos 2l\right]$$
$$- \frac{7}{2}\frac{g_2 e}{a^2} t - \frac{2g_2}{a^2 n}\sin l - \frac{5g_2 e}{4a^2 n}\sin 2l + \cdots$$

$$e\delta\varpi = \frac{g_1}{2}\left[\frac{n'}{n} - 1\right]\frac{\cos(3l - 2l')}{6n - 4n'} - \frac{3g_1}{2}\left[\frac{n'}{n} - 1\right]\frac{\cos(l - 2l')}{2n - 4n'}$$
$$- \frac{g_1}{n}\left[\frac{n'}{n} - 1\right]\cos l - \frac{2}{3}g_1 t - \frac{g_1 e}{12n}\cos l - \frac{g_1 e}{3n}\sin l$$
$$- \frac{K'}{m_2}[\sin l + e\sin 2l] + \frac{2g_2}{a^2 n}\cos l + \frac{g_2 e}{2a^2 n}\cos 2l,$$

$$\delta\epsilon_1 = \frac{g_1}{2n}\cos 2(l - l') - \frac{2g_1 e}{3n}\cos l - \frac{2nK'}{m_2}\left[t + \frac{e}{n}\sin l\right] - \frac{8g_2 e}{a^2 n}\cos l.$$

The complete integral may be obtained by adding the two sets of expressions for δa, δe, $e\delta\varpi$ and $\delta\epsilon_1$ respectively. There is a constant appearing through the mean motion which should be added to δa. This has been omitted because it does not directly concern the present problem.

3. Consider now the effect on the motion of the particle. It is evident that it will suffer solar eclipse during a portion of each revolution. During the time it is in Jupiter's shadow the radiation pressure from the Sun will be cut off. We shall first determine the effect of this eclipse on the semi major axis and eccentricity.

Denoting the angle which the intersection of the shadow and the orbit subtends at the center of Jupiter by 2β we have, when the particle enters the shadow,

$$l - l' = \pi - \beta,$$

and at the time it leaves the shadow, taking the orbit to be circular,

$$l - l' = \pi + \beta.$$

Now the change in r is given by

$$\delta r = \delta a - \delta e\cos l + e\delta\varpi\sin l,$$

neglecting products of the changes with e. From this we find the change in the value of r due to the eclipse. Since the longitude of the perijove is

constantly increasing, we can neglect, in making the substitution, all functions involving angles other than $\omega - \varpi - \omega'$. This gives us

$$\delta r = \delta a = \left[\frac{ag_1}{2n} \sin 2(l - l') \right]_{\pi+\beta}^{\pi-\beta} = -\frac{ag_1}{n} \sin 2\beta. \tag{1}$$

Hence the particle is drawn in toward the attracting center because of the eclipse.

If for an approximation we call n a constant, we have

$$\beta = \beta' \frac{1}{1 - \frac{n'}{n}}, \tag{2}$$

where β' is the angle between the line joining the sun and Jupiter produced, and the line joining the particle and Jupiter at the instant that the particle enters the shadow and where the particle revolves in the same direction as the Sun. If it revolves in the opposite direction,

$$\beta = \beta' \frac{1}{1 + \frac{n'}{n}}. \tag{3}$$

If t_1 is the time it takes the particle to make the remainder of the revolution, we have for direct motion

$$t_1 = \frac{2\pi}{n} - \frac{2\beta'}{n} \frac{1}{1 - \frac{n'}{n}},$$

and

$$t_1 = \frac{2\pi}{n} - \frac{2\beta'}{n} \frac{1}{1 + \frac{n'}{n}}$$

for retrograde motion.

During this time the particle is again drawn in the additional distance

$$\left[\frac{5}{3} ag_1 - ag_1 \frac{n'}{n} \right] t_1 = \frac{aK}{VR^2} \left[\frac{5}{3} - \frac{n'}{n} \right] \left[\frac{2\pi}{n} - \frac{2\beta'}{n} \frac{1}{1 - \frac{n'}{n}} \right] \tag{4}$$

for direct motion and

$$\frac{aK}{VR^2} \left[\frac{5}{3} - \frac{n'}{n} \right] \left[\frac{2\pi}{n} - \frac{2\beta}{n} \frac{1}{1 + \frac{n'}{n}} \right] \tag{5}$$

for retrograde motion.

In addition to this throughout the whole revolution the radiation from

Jupiter is affecting the particle causing it to fall in the additional distance

$$\frac{4K'\pi}{3Van}. \tag{6}$$

Thus the change in the semi major axis per revolution will be approximately

$$-\frac{aK}{VR^2n}\left\{\sin 2\beta' + 2\beta'\frac{n'}{n}\cos 2\beta' + \left[\frac{5}{3}-\frac{n'}{n}\right]\left[2\pi - 2\beta'\frac{1}{1-\frac{n'}{n}}\right]\right\} - \frac{4K'\pi}{3Van} \tag{7}$$

if the motion is direct, and

$$-\frac{aK}{VR^2n}\left\{\sin 2\beta - 2\beta'\frac{n'}{n}\cos 2\beta' + \left[\frac{5}{3}-\frac{n'}{n}\right]\left[2\pi - 2\beta'\frac{1}{1+\frac{n'}{n}}\right]\right\} - \frac{4K'\pi}{3Van} \tag{8}$$

if the motion is retrograde.

Treating similarly the expression for δe we get the change in eccentricity per revolution. This is

$$-\frac{K}{VR^2n}\left\{\frac{e}{4}\left[\frac{n'+2n}{n-n'}\right]\left[\sin 2\beta' + 2\beta'\frac{n'}{n}\cos 2\beta'\right] + \frac{5e}{4}\frac{n'}{n}\left[2\pi - 2\beta'\frac{1}{1-\frac{n'}{n}}\right]\right\} - \frac{7K'e\pi}{3Va^2n}$$

when the particle revolves with the sun and

$$-\frac{K}{VR^2n}\left\{\frac{e}{4}\left[\frac{n'+2n}{n-n'}\right]\left[\sin 2\beta' - 2\beta'\frac{n'}{n}\cos 2\beta'\right] + \frac{5e}{4}\frac{n'}{n}\left[2\pi - 2\beta'\frac{1}{1+\frac{n'}{n}}\right]\right\} - \frac{7K'e\pi}{3Va^2n},$$

when it revolves in a retrograde direction.

Finally the particle will not make a number of revolutions N greater than

$$N = \frac{3Vna^2R^2}{\pi\left(5Ka^2 + 4K'R^2 - 11\frac{n'}{n}Ka^2\right)} \tag{9}$$

for direct motion and

$$N_R = \frac{3Vna^2R^2}{\pi\left(5Ka^2 + 4K'R^2 + 5\frac{n'}{n}Ka^2\right)} \tag{10}$$

for the retrograde motion before it reaches the surface of Jupiter. The The difference of these will be

$$N - N_R = \frac{3Vna^2R^2 \cdot 16\frac{n'}{n}Ka^2}{\pi(5Ka^2 + 4K'R^2)\left(5Ka^2 + 4K'R^2 - 6\frac{n'}{n}Ka^2\right)}. \quad (11)$$

4. As a numerical example, we shall assume the following data.

Velocity of light, $V = 3 \times 10^{10}$,
Mass of the Sun, $m_1 = 1.29 \times 10^{26}$,
Mass of Jupiter, $m_2 = 1.29 \times 10^{23}$,
Distance of Jupiter from the Sun, $R = 7.8 \times 10^{13}$.

All of these are in the C.G.S. units.* The value of K may be determined from the known pressure of light at the earth's surface. If the particle has a density about that of the earth and a radius of about .01 cm., K has the value 1.76×10^{23}.* We also assume

Constant of radiation of Jupiter, $K' = K \times 10^{-4}$,
Distance of the particle from Jupiter at an apse, $a = \frac{1}{2}R \times 10^{-2}$.

Then if we suppose the shadow cast by Jupiter is a cylinder whose base is a great circle of Jupiter, β' will be equal to .0178 radians or a little more than 1°. Then it will be found from (1) with (2) or (3) that, due to the eclipse, the direct particle will fall in a distance 8.98 cm. while the retrograde particle will fall in a distance 9.40 cm. It may be noted that these values increase to a maximum when the particle has fallen in so that β' equals $\pi/4$ and then decreases to nearly 0 when the particle skims the surface of Jupiter. During the rest of the revolution equations (4), (5) and (6) show that the direct particle falls in a distance .688 km., while the retrograde particle falls in a distance .689 km. These will be approximately the totals as the eclipse effect is not quite appreciable at this distance.

Treating the expression for the change in the eccentricity in the same way we find the eclipse change for the direct particle is $.122 \times 10^{-10}e$ while that for the retrograde particle is $.117 \times 10^{-10}e$. During the rest of the revolution the change for the direct particle will be $19.15 \times 10^{-9}e$ which is approximately that for the retrograde particle. Thus the total change will be $19.16 \times 10^{-9}e$ for both particles.

To find the greatest number of revolutions we will start the particle a

* Wilson, Annals of Math., 2d ser., vol. 7–8 (1905–07), p. 148.

little farther out from Jupiter and let

$$\text{Distance from Jupiter, } a = R \times 10^{-2}.$$

Then the number of revolutions which is found from (9) or (10) probably will not be more than 5.87×10^7 for the direct and 5.54×10^7 for the retrograde particle. The difference between these, given by (11), is 3.3×10^6. This indicates that the retrograde particle is drawn in much faster, a fact which can be verified in general by comparing equations (7) and (8).

NON-HOMOGENEOUS LINEAR EQUATIONS IN INFINITELY MANY UNKNOWNS.

By A. J. Pell.

Introduction. In this paper we consider the solution of the system of non-homogeneous linear equations in infinitely many unknowns,

$$\sum_{k=1}^{\infty} a_{ik}x_k = c_i \qquad (i = 1, 2, 3, \cdots), \tag{1}$$

with the object of subjecting the matrix (a_{ik}) to as few restrictions as possible. We are concerned, in particular, with the derivation of conditions to be imposed on the constants $\{c_i\}$ in order that a solution $\{x_i\}$ of (1) exist, and the determination of the character of the solution.

From the articles dealing with linear equations in infinitely many unknowns we cite only those results which have some bearing on the problem stated above.*

Assuming that the matrix (a_{ik}) is such that $\Sigma_{k=1}^{\infty} a_{ik}^2$ converge for every i, E. Schmidt† derives the necessary and sufficient condition, to be imposed on the $\{c_i\}$, for the existence of a solution $\{x_i\}$ of *finite norm*. On account of the complexity of the processes involved, these conditions are difficult to apply in particular cases.

The method of Kötteritzsch‡ consists in the reduction of a general system of equations to one for which

$$a_{ii} \neq 0, \qquad a_{ik} = 0, \qquad i > k, \tag{2}$$

and in the formal expression of the solution $\{x_i\}$ in terms of the $\{c_i\}$ by means of a reciprocal matrix. Helge von Koch§ and Carmichael‖ have

* Hilbert and Toeplitz consider the following problem: the determination of conditions to be imposed on the matrix, already restricted to be limited or even more, in order that a solution of *finite norm* exist *whenever* the constants $\{c_i\}$ are of *finite norm*. In some articles dealing with infinite determinants Helge von Koch considers a similar problem.

† Rendiconti del Circolo Matematico di Palermo, XXV, 1908, pp. 53–77. See also Bôcher and Brand, *Annals of Mathematics*, 1912, pp. 167–186.

‡ "Über die Auflösung eines Systems von unendlich vielen linearen Gleichungen," Zeitschrift für Mathematik und Physik, t. XV, 1870, pp. 1–15, 229–268. See also E. Goldschmidt, Über die numerische Verwendbarkeit der Methoden zur Auflösung unendlich vieler linearer Gleichungen, Dissertation, Würzburg, 1912, §§ 2, 5; and F. Riesz, Les Systèmes d'Équations linéaires a une Infinité d'Inconnues, 1913, p. 11, 12.

§ "On the regular and irregular solutions of some infinite systems of linear equations," Proceedings Fifth International Congress of mathematicians, vol. 1, pp. 352–366.

‖ "On non-homogeneous equations with an infinite number of variables," January, 1914, pp. 13–20.

derived conditions under which this formal expression yields a solution of the system of equations.

To the conditions (2) Helge von Koch adds

$$a_{ii} = 1, \qquad |a_{ik}| < T^{k-i},$$

and shows that

$$\overline{\lim_{i=\infty}} \sqrt[i]{c_i} < \frac{1}{k}$$

is necessary and sufficient for the existence of a unique solution $\{x_i\}$ such that

$$\overline{\lim_{i=\infty}} \sqrt[i]{x^i} < \frac{1}{k}.$$

Carmichael adds to (2) only the condition that

$$a_{ii} = 1,$$

and his sufficient conditions to be imposed on the $\{c_i\}$ are broader than those of Helge von Koch, but are more complicated in form. No information is given about the character of the solutions, and for the general system of equations there is no indication of the nature of the right-hand side for which solutions exist.

The conditions which we derive for the applicability of the Kösseritzsch method under (2) are simpler in form and more general than those of Carmichael. We also gain some information about the character of the right-hand side and the corresponding solutions for both the general and reduced systems of equations.

1. Definitions and transformation of matrices. A sequence of constants $\{c_i\}$ is of *finite norm* if $\Sigma_{i=1}^{\infty} c_i^2$ converges.

A matrix (a_{ik}) is *limited** if there exists a positive quantity M independent of $\{x_i\}$ and $\{y_i\}$ and n such that

$$\left| \sum_{i,\,k=1}^{n} a_{ik} x_i y_k \right| \leqq M \sqrt{\sum_{i=1}^{n} x_i^2 \sum_{k=1}^{n} y_k^2}$$

for all sequences $\{x_i\}$ and $\{y_i\}$ of finite norm.

For convenience of later reference we state here some well-known properties of limited matrices.

(*A*) If the matrix (a_{ik}) is limited,† the transformation

$$y_i = \sum_{k=1}^{\infty} a_{ik} x_k$$

* Hellinger-Toeplitz, "Theorie der unendlichen Matrizen," Mathematische Annalen, 69, 1910, p. 296.

† Hilbert, Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen, p. 128.

transforms every sequence $\{x_i\}$ of finite norm into a sequence $\{y_i\}$ of finite norm.

(*B*) If the matrix (a_{ik}) is limited,* and the sequences $\{x_i\}$ and $\{y_i\}$ are of finite norm,

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} a_{ik}x_iy_k = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}x_iy_k.$$

(*C*) The matrix (a_{ik}) is limited† if the sequence $\{\Sigma_{k=1}^{\infty} a_{ik}x_k\}$ is of finite norm for every sequence $\{x_i\}$ of finite norm.

For the work in the sequel we need the following lemmas concerning the transformation of matrices.

Lemma 1. *If (a_{ik}) is a matrix such that the sequence $\{a_{ik}\}$ is not of finite norm for every i, and if*

$$|\lambda_k| > \frac{\text{max.}\,|a_{ik}|\ (i = 1, 2, \cdots k)}{|\epsilon_k|}, \tag{3}$$

where $\{\epsilon_k\}$ is any sequence of finite norm, the matrix (a_{ik}/λ_k) has the property that the sequence $\{a_{ik}/\lambda_k\}$ is of finite norm for every i.

The lemma follows immediately from the term by term inequality

$$\sum_{k=1}^{\infty}\frac{a_{ik}^2}{\lambda_k^2} \leqq \sum_{k=1}^{i}\frac{a_{ik}^2}{\lambda_k^2} + \sum_{k=i+1}^{\infty}\epsilon_k^2.$$

Obviously it is sufficient to take

$$|\lambda_k| = \frac{\sqrt{\sum_{i=1}^{k} a_{ik}^2}}{|\epsilon_k|},$$

instead of the value given in (3),

Lemma 2. If (a_{ik}) *is an unlimited matrix such that* $a_{ik} = 0$ *for* $i > k$, *and if*

$$|\lambda_k| > \frac{\text{max.}\,|a_{ik}|\ (i = 1, 2, \cdots k)}{|\epsilon_k|},$$

where $\{\epsilon_k\}$ is any sequence of finite norm, the matrix (a_{ik}/λ_k) is limited.

This lemma is a consequence of the inequality

$$\sum_{k=1}^{\infty}\left(\sum_{i=1}^{k}\frac{a_{ik}}{\lambda_k}x_i\right)^2 \leqq \sum_{k=1}^{\infty}\epsilon_k^2\sum_{i=1}^{k}x_i^2 \leqq \sum_{k=1}^{\infty}\epsilon_k^2\sum_{i=1}^{\infty}x_i^2,$$

and the property (*C*).

2. **Special systems of linear equations.**—Let

$$\sum_{k=1}^{\infty} a_{ik}x_k = c_i \qquad (i = 1, 2, 3, \cdots) \tag{4}$$

* Hilbert, l. c., p. 129.

† Hellinger-Toeplitz, l. c., § 10.

be a system of linear equations with the following condition on the matrix (a_{ik})

$$a_{ii} \neq 0, \qquad a_{ik} = 0 \qquad i > k. \tag{5}$$

Let the matrix (b_{ik}) be defined by

$$b_{ik} = 0 \qquad i \neq k,$$
$$b_{ik} = \frac{\bar{b}_{ik}}{a_{11}a_{22} \cdots a_{ii}} \qquad i \leqq k, \tag{6}$$

where $\bar{b}_{ik}$ is the value of the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22} & \cdots & a_{2k} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & a_{ik} \end{vmatrix}$$

with $0, 0, \cdots 1$ substituted in the kth column. Then the matrix (b_{ik}) is a reciprocal matrix of (a_{ik}), or

$$\Sigma a_{i\alpha} b_{k\alpha} = \begin{cases} 0 & i \neq k. \\ 1 & i = k. \end{cases} \tag{7}$$

THEOREM 1. *If the sequences $\{\lambda_k\}$ and $\{\mu_j\}$ are such that the sequence $\{a_{ik}/\lambda_k\}$ is of finite norm for every i, and the matrix $(\lambda_k b_{ik}/\mu_i)$ is limited, then for every sequence $\{c_i\}$ such that $\{\mu_i c_i\}$ is of finite norm, the system of equations (4) has a solution $\{x_i\}$ such that $\{\lambda_i x_i\}$ is of finite norm, and the solution is given by*

$$x_i = \sum_{k=1}^{\infty} b_{ik} c_k \qquad (i = 1, 2, 3, \cdots).$$

If the $\{a_{ik}\}$ is not of finite norm for every i, the sequence $\{\lambda_i\}$ can be chosen as in Lemma 1; if the $\{a_{ik}\}$ is of finite norm for every i, take $\lambda_k = 1$, or as a decreasing sequence in some cases. The matrix $(\lambda_i b_{ki})$ satisfies the conditions of Lemma (2), and the constants $\{\mu_i\}$ can be chosen so that the matrix $(\lambda_i b_{ki}/\mu_k)$ is limited. Applying next property (B) § 1, and the relation (7), we obtain

$$\sum_{\alpha=1}^{\infty} \sum_{k=1}^{\infty} \frac{a_{i\alpha}}{\lambda_\alpha} \left(\frac{\lambda_\alpha b_{k\alpha}}{\mu_k} \right) \mu_k c_k = \sum_{k=1}^{\infty} \sum_{\alpha=1}^{\infty} \frac{a_{i\alpha}}{\lambda_\alpha} \left(\frac{\lambda_\alpha b_{k\alpha}}{\mu_k} \right) \mu_k c_k = c_i, \tag{8}$$

if the sequence $\{c_i\}$ is such that $\{\mu_i c_i\}$ is of finite norm. From (8) we have that a solution of (4) is given by

$$x_i = \sum_{k=1}^{\infty} b_{ik} c_k \qquad (i = 1, 2, 3, \cdots)$$

and from property (A) § 1, the sequence $\{\lambda_i x_i\}$ is of finite norm.

Lemma (1) gives us a possible value for $\{\lambda_k\}$ in terms of the elements of the matrix (a_{ik}), if $\{a_{ik}\}$ is not of finite norm for every i, and Lemma (2) gives it for the $\{\mu_i\}$ in terms of the elements of the matrix (b_{ik}). It is desirable to have a value for $\{\mu_i\}$ in terms of a_{ik}, so that it will not be necessary to compute the b_{ik}. By the Hadamard theorem on determinants,

$$|b_{ik}| \leqq \frac{\sqrt{\prod_{k=1}^{i} \sum_{i=1}^{k} a_{ik}^2}}{|a_{11}| \cdot |a_{22}| \cdots |a_{ii}|}. \tag{9}$$

Using this together with Lemma 2 we have

Corollary. The system of equations (4) *has a solution* $\{x_i\}$ *such that* $\{\lambda_i x_i\}$ *is of finite norm for every* $\{c_i\}$ *such that* $\{\mu_i c_i\}$ *is of finite norm, where*

$$|\lambda_k| = \frac{\text{max. } |a_{ik}|}{|\epsilon_k|} \qquad (i = 1, 2, \cdots k), \tag{10}$$

if $\{a_{ik}\}$ *is not of finite norm for every* i, *and*

$$0 < |\lambda_k| \leqq 1, \tag{10'}$$

if $\{a_{ik}\}$ *is of finite norm for every* i, *and*

$$|\mu_i| = \frac{\sqrt{\prod_{k=1}^{i} \sum_{j=1}^{k} a_{jk}^2}}{|a_{11}| \cdot |a_{22}| \cdots |a_{ii}|} \cdot \frac{\text{max. } |\lambda_k|}{|\epsilon_i|} \qquad (k = 1, 2, \cdots i), \tag{11}$$

$\{\epsilon_i\}$ *being any sequence of finite norm.*

If $\{\lambda_i\}$ and $\{\mu_i\}$ are given by (10) and (11) respectively the sequence $\{c_i\}$ satisfying the condition of the corollary, will be such that $\Sigma_{i=1}^{\infty} |c_i|$ converges, if given by (10′) and (11) respectively, the sequence $\{c_i\}$ is of finite norm.

The same method of proof applies for the following theorem and corollary.

THEOREM 2. *If* $\{\bar{\lambda}_i\}$ *and* $\{\bar{\mu}_i\}$ *are sequences of constants such that* $\{b_{ik}/\bar{\mu}_i\}$ *is of finite norm for every* i, *and the matrix* $(\bar{\mu}_i a_{ik}/\lambda_k)$ *is a limited matrix, then any solution* $\{x_i\}$ *of the system of equations* (4) *such that* $\{\bar{\lambda}_i x_i\}$ *is of finite norm, is given by*

$$x_i = \sum_{k=1}^{\infty} b_{ik} c_k \qquad (i = 1, 2, 3, \cdots),$$

and the $\{c_i\}$ *are such that the sequence* $\{\mu_i c_i\}$ *is of finite norm.*

Corollary. If

$$|\bar{\mu}_i| = \frac{\sqrt{\prod_{k=1}^{i} \sum_{i=1}^{k} a_{ik}^2}}{|a_{11}| \cdot |a_{22}| \cdots |a_{ii}|} \frac{1}{\epsilon_i}, \tag{12}$$

and

$$|\bar{\lambda}_k| = \frac{\text{max. } |a_{ik}|\,|\mu_i|}{|\epsilon_k|} \qquad (i = 1, 2, \cdots k)$$

where $\{\epsilon_i\}$ *is any sequence of finite norm, then any solution* $\{x_i\}$ *of the system of equations* (4) *such that the sequence* $\{\lambda_i x_i\}$ *is of finite norm, is given by*

$$x_i = \sum_{k=1}^{\infty} b_{ik} c_k \qquad (i = 1, 2, 3, \cdots),$$

and the $\{c_i\}$ *is such that the sequence* $\{\mu_i c_i\}$ *is of finite norm.*

3. General system of equations. Let

$$\sum_{k=1}^{\infty} A_{ik} x_k = C_i \qquad (i = 1, 2, 3, \cdots) \tag{14}$$

be a system of linear equations for which the determinants

$$\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1i} \\ A_{21} & A_{22} & \cdots & A_{2i} \\ \cdot & \cdot & \cdot & \cdot \\ A_{i1} & A_{i2} & \cdots & A_{ii} \end{vmatrix} \neq 0 \qquad (i = 1, 2, 3, \cdots).$$

Then it can be reduced to the system (4) and (5), and the relations between the matrices and right-hand sides of the equations have the form

$$A_{ik} = \sum_{k=1}^{i} \xi_{i\alpha} a_{\alpha k} \qquad i \leqq k \ (k = 1, 2, 3, \cdots), \tag{15}$$

$$C_i = \sum_{k=1}^{i} \xi_{i\alpha} c_\alpha \qquad (i = 1, 2, 3, \cdots). \tag{16}$$

The coefficients ξ_{ik} are not uniquely determined as each row contains a multiplicative constant. By Lemma 2 these constants can be chosen so that the matrix (ξ_{ik}) (where $\xi_{ik} = 0$ for $k > i$) is limited. Therefore by property (A) §1, the sequence $\{C_i\}$ is of finite norm if the corresponding sequence $\{c_i\}$ is also of finite norm. From the form of (15) it follows that, if $\{a_{ik}/\lambda_k\}$ is of finite norm for every i, then $\{A_{ik}/\lambda_k\}$ is also of finite norm for every i, and vice-versa. Applying Lemma 1 and the corollary to Theorem 1, we have

THEOREM 3. *If*

$$|\lambda_i| > \frac{\text{max. } A_{ik} \ (i = 1, 2, \cdots k)}{\epsilon_i}$$

and

$$|\mu_i| = \frac{\sqrt{\prod_{k=1}^{i} \sum_{j=1}^{k} a_{jk}^2}}{|a_{11}| \cdot |a_{22}| \cdots |a_{ii}|} \cdot \frac{\text{max. } |\lambda_k| \ (k = 1, 2, \cdots i)}{|\epsilon_i|},$$

where $\{\epsilon_i\}$ *is any sequence of finite norm, and if* $\{C_i\}$ *has the form* (16) *where* $\{c_i\}$ *is such that* $\{\mu_i c_i\}$ *is of finite norm (and therefore* $\{C_i\}$ *is of finite norm), the system of equations* (14) *has a solution* $\{x_i\}$ *such that* $\{\lambda_i x_i\}$ *is of finite norm.*

March, 1914.

THE INSCRIBED AND CIRCUMSCRIBED SQUARES OF A QUADRILATERAL AND THEIR SIGNIFICANCE IN KINEMATIC GEOMETRY.

By C. M. Hebbert.

Professor Emch has shown that in any closed convex curve at least one square may be inscribed* and that about such a curve at least one square may be circumscribed. Moreover, the results may be extended to convex polygons formed by analytic arcs.

This investigation leads to methods for actually constructing the inscribed squares of all regular polygons and any quadrilateral and the circumscribed square of any quadrangle. By inscribed square is meant one whose vertices lie on four sides of the figure and by circumscribed square is meant a square each of whose ~~vertices~~ sides passes through a vertex of the quadrangle. Because of the number of interesting propositions of kinematic geometry resulting from the solution of these problems for the quadrilateral and quadrangle, I shall confine myself to their discussion.

Theorem I. *In every quadrilateral may be inscribed at least one square having a vertex on each of the four sides. If there is more than one such square in a given quadrilateral, there is an infinite number.*#

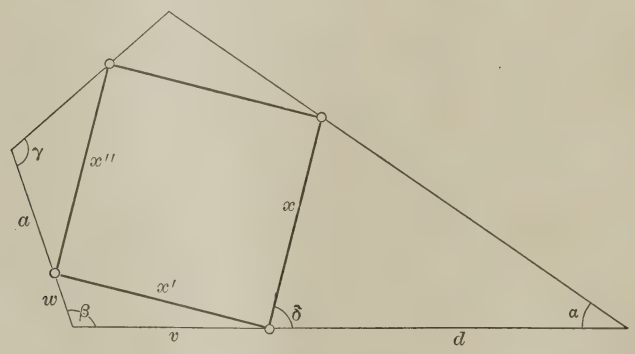


Fig. 1.

For let x be the side of a square having its vertices on the four sides of a given quadrilateral and let δ be the angle made by the side x and the side d of the quadrilateral. Also let α, β, γ be the angles of the quadrilateral as indicated and let a be the side included by the angles β and γ. v and w are the segments cut off from d and a, respectively, by corresponding vertices of the square.

* "Some Properties of Closed Convex Curves in a Plane," American Journal of Mathematics, vol. 35, pp. 407–412.

For a geometrical interpretation[38] see article by Miss Raku Makita, Tohoku math. Jour. Vol. 9, pp. 77–79

Then making use of the law of sines and the conditions that $x = x' = x''$ and $x \perp x'$ and $x' \perp x''$, we get

$$\tan \delta = \frac{d(\csc \beta + \cos \beta + \cot \gamma \sin \beta) - a(1 + \cot \beta)}{d(\cot \gamma \cos \beta - \sin \beta) + a(1 + \cot \alpha)}.$$

This equation shows that in general there is always one real value for δ in a given quadrilateral. The equation can have more than one solution if and only if the right-hand member is 0/0 and then there are an infinite number of solutions, according to a theorem stated by A. Hurwitz:*

"If it is possible in special cases to get more than n solutions for a problem which in general has n solutions determined by the roots of an equation of the nth degree, then in these special cases there are an infinite number of solutions."

Since V and W are linear in δ, every point of the side of a quadrilateral in which an infinite number of squares can be inscribed is the vertex of one and only one inscribed square.

THEOREM II. *If three vertices of a variable square move along three fixed straight lines, the fourth vertex moves along a fixed straight line and the centers are collinear.*

For if each of two squares has three of its vertices on three fixed lines, the line joining the fourth vertices will form with the three fixed lines a quadrilateral in which two squares are inscribed. Hence in the quadrilateral so formed an infinite number of squares may be inscribed and the fourth side so determined is the line along which the fourth vertex moves.

Differentiation and division of the linear expressions for V and W in Fig. 1, gives dV/dW = a constant, i. e. the ratio of the distances through which these two vertices move is constant. Similarly for any pair of vertices.

Let ABC and $A'B'C'$ (Fig. 2) be three positions of opposite vertices. Then AA', BB', CC' will be diagonals. For the sake of simplicity, assume that $AB = BC$ and $A'B' = B'C'$. Let F, G, H be the mid-points of AA', BB', CC' respectively.

Then $FBHB'$ is a parallelogram and since the diagonal FH bisects the diagonal BB', F, G, and H are collinear. Also $FG = GH$.

Kinematically, these results may be expressed as follows:

THEOREM III. *If two vertices of a variable square move with constant velocities along two fixed straight lines, the other two vertices move with constant velocities along two fixed lines and the center moves with constant velocity along a straight line.*

Some special cases of this theorem have been stated by E. Study, who proved them by means of the " real pictures " of imaginary points.†

* Math. Annalen, vol. XV, p. 8.

† E. Study, Vorlesungen über ausgewählte Gegenstände der Geometrie. Erstes Heft: Ebene analytische Kurven und zu ihnen gehörige Abbildungen, p. 16.

With the aid of these theorems we can easily prove the correctness of the following construction, due to A. Emch, for the square inscribed in a given

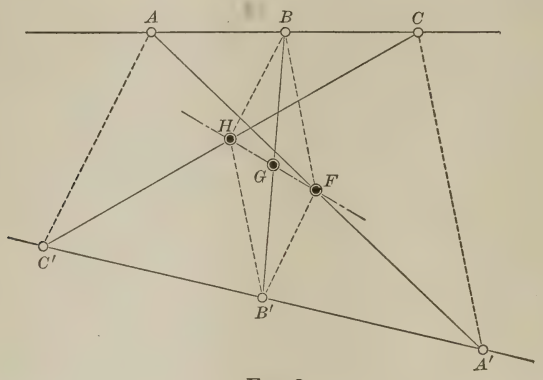


FIG. 2.

quadrilateral. Given quadrilateral $ABCD$. Through any point, E, on AB draw two perpendicular lines cutting BC and AD in F and G re-

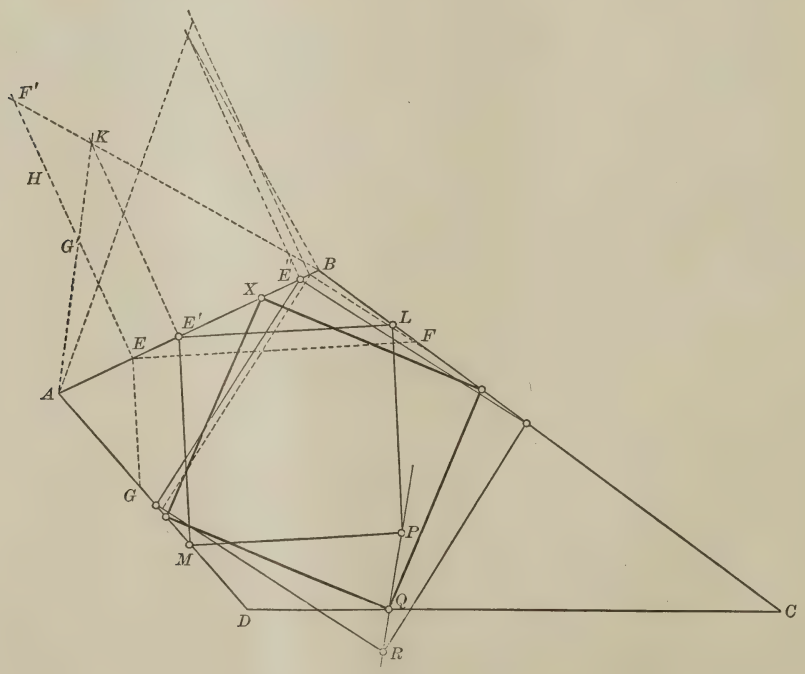


FIG. 3.

spectively. Draw EH perpendicular to AB and on EH lay off EF' and EG' equal to EF and EG, respectively. From K, the sect-point of BF' and AG' draw KE' perpendicular to AB. Then $E'L$ and $E'M$ drawn parallel

to EF and EG, respectively, are equal and determine a square having three vertices on three sides of the given quadrilateral.

Repetition of this process gives a second square having three vertices on the same three sides of the given quadrilateral.

Now if we think of this second square as being obtained by allowing the vertices of the first square to move along the three straight lines DA, AB, and BC, while the sides vary in length, we see that by Theorem II, the fourth vertex must have moved along the fixed straight line PR. Then the sect-point, Q, of PR and DC will be a vertex of the square having its four vertices on the four sides of the quadrilateral. The other three vertices can be easily determined since by Theorem III, $PQ/PR = E'X/E'E''$, etc.

Since the vertices of the variable square in Theorems II and III move with constant velocities along the four sides of the quadrilateral, the sides

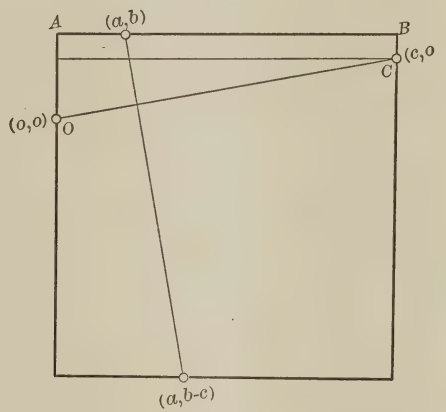


FIG. 4.

of the variable square envelope four parabolas. Consideration of the similar triangles formed by joining the focus of one of the parabolas to the vertices of two of the squares gives

THEOREM III*a*. *The four sides of the variable square whose four vertices move along four fixed straight lines envelope four confocal parabolas whose directrices form a square.*

THEOREM IV. *If three sides of a variable square pass through three fixed points, the fourth side passes through a fixed point.*

Given three points (0, 0), (a, b) and $(c, 0)$ and through these points the lines $y = nx$, $y = n(x - c)$, and $n(y - b) + x - a = 0$.

Since AB is $\perp$ to AO and BC, $AB = OC \sin AOB = cn/\sqrt{1 + n^2}$.

Now in order for our fourth line to form a square with the three given lines, it must be at the distance $\pm cn/\sqrt{1 + n^2}$ from AB and be parallel to AB.

The distance of any point $(x_1\ y_1)$ from the line $n(y-b)+(x-a)=0$ is

$$\frac{ny_1+x_1-nb-a}{\sqrt{1+n^2}}.$$

Since we want (x_1y_1) to be at distance $\pm cn/\sqrt{1+n^2}$ from $n(y-b)+(x-a)=0$, we have as equation of locus of (x_1y_1)

$$\frac{ny_1+x_1-nb-a}{\sqrt{1+n^2}}=\frac{\pm cn}{\sqrt{1+n^2}}$$

or,

$$n(y_1-b\pm c)+(x_1-a)=0$$

which may be written

$$\frac{y_1-(b\pm c)}{x_1-a}=-\frac{1}{n}.$$

This last form of the equation shows that the fourth side of the square through the three given points always passes through the point $(a,\ b\pm c)$.

It will be observed that the points $(a,\ b)$ and $(a,\ b\pm c)$ are on the line $x=a$ which is $\perp$ to the x axis, and that the distance between these two points is equal to c.

Hence we have

Corollary 1. *A necessary and sufficient condition that it be possible to circumscribe an infinite number of squares about a given quadrangle is that the diagonals be perpendicular and equal.*

Therefore, to construct the circumscribed square of the quadrangle $ABCD$, draw $BE\perp$ and equal to AC. Through B draw a parallel to ED and through A and C draw perpendiculars to ED. These three lines and ED form the required square.*

The kinematic expression of theorem IV is:

Theorem V. *If one side of a variable square turns about a fixed point while the extremities of this side move with constant velocities on two circles which pass through the fixed point, then the other three sides of the square turn about fixed points and the two remaining vertices of the square move with constant velocities on two circles which intersect each other and the two given circles in the remaining fixed points. The centers of the four circles form a square. The center of the variable square moves on a circle which passes through the sect-point and the mid-points of the diagonals joining the four fixed points. The velocities of the vertices and the center of the variable square are proportional to the radii of the corresponding circles, i. e. the angular velocities of vertices and center are constant.*†

* For another construction, see Journal für die reine und angewandte Mathematik, vol. XXXIV, p. 281.

† See Study, loc. cit., and A. Mannheim: Principes et Développements de Géométrie Cinematique, pp. 14–17.

A NOTE ON SYMMETRIC MATRICES.*

By Gilbert Ames Bliss

In a recent paper Dickson† has given a very concise and elementary proof of the well-known fact that for a symmetric matrix of rank r at least one principal minor of order r is not zero. Wedderburn‡ deduced the same result for matrixes with real elements in proving, by a less elementary though very concise method, that the sum of the principal minors does not vanish under the same circumstances. The purpose of this note is to exhibit a theorem which includes both of the results just mentioned, and throws further light on the structure of a symmetric matrix.

Suppose that the order of a square matrix A is m, and let the combinations of r letters selected out of the set $(1, 2, \cdots, m)$ be denoted by the symbols P_i $(i = 1, 2, \cdots, n)$. The symbol A_{ik} will represent the minor of A of order r with rows corresponding to the numbers P_i and columns to the numbers P_k. The matrix (A_{ik}) is evidently symmetric if the original matrix A has this property.

For a symmetric matrix of rank r there exists a multiplier $M \neq 0$ and a set of numbers X_i $(i = 1, 2, \cdots, n)$ not all zero such that

$$A_{ij} = MX_iX_j \qquad (i, j = 1, 2, \cdots, n). \tag{1}$$

The justification of these statements depends upon the known fact, a proof of which is given below, that the ratios $A_{i1} : A_{i2} : \cdots : A_{in}$ $(i = 1, 2, \cdots, n)$ for a matrix of rank r or less are all the same.§ In other words there exists a set of numbers $X_1, X_2, \cdots, X_n$ not all zero, with a set of multipliers M_i $(i = 1, 2, \cdots, n)$ such that

$$A_{i1} = M_iX_1, \quad A_{i2} = M_iX_2, \quad \cdots, \quad A_{in} = M_iX_n \quad (i = 1, 2, \cdots, n).$$

Any row $A_{i1}, A_{i2}, \cdots, A_{in}$ with elements not all zero would be a set of numbers X with these properties. Suppose that X_k is different from zero. Then M is uniquely determined by the condition $A_{kk} = MX_k^2$, and it follows that the multiplier M_k of the kth row has the value MX_k, while the elements A_{ki} of that row have the values

$$A_{ki} = MX_kX_i = A_{ik} \qquad (i = 1, 2, \cdots, n).$$

* Presented to The American Mathematical Society, April, 1914.

† *Annals of Mathematics*, vol. 15 (1913), page 27.

‡ Page 29 of the same journal.

§ Kowalewski, Einführung in die Determinantentheorie, page 122.

From the fact that this is also an expression for A_{ik} it follows at once that the multiplier M_i for the ith row has the value MX_i. The constant M can not be zero; otherwise the matrix would be of rank less than r.

If a matrix A, symmetric or not, has its ratios $A_{i1} : A_{i2} : \cdots : A_{in}$ identical, the same will be true after a row has been multiplied by a constant and added to another, and vice versa. The rank is also unchanged by this transformation. In proving that the ratios $A_{i1} : A_{i2} : \cdots : A_{in}$ are all equal, in the sense described above, for a matrix of rank r, it may be supposed without loss of generality that A_{11} is the principal minor of order r in the upper left-hand corner of A, and that it is different from zero. If the first r rows of A are multiplied by suitable constants and added to the remaining ones, it can be brought about that all the elements of A below A_{11} are zero. Since the resulting matrix has still rank r, it follows that all the other elements of the last $m - r$ rows must also be zero. But in that case the rows of elements $A_{i1}, A_{i2}, \cdots, A_{in}$ are all zero except when $i = 1$, and all the rows have the same ratio.

Since at least one X_k is different from zero it follows from the italicized theorem above that at least one principal minor is different from zero. If any minor A_{ik} of order r, which is not principal, is different from zero, then at least two principal minors do not vanish, namely, A_{ii} and A_{kk}. The sum of the principal minors of order r is evidently

$$\Sigma A_{ii} = M\Sigma X_i^2.$$

For the case of a matrix of real elements this is evidently different from zero, and all the principal minors which are different from zero have the same sign.

I am indebted to Mr. Meyer Gaba for the suggestion that the methods used above should also be applicable to Hermitian matrices for which any pair of elements symmetric with respect to the principal diagonal are conjugate imaginaries. This is in fact the case, and the minors of order r of a Hermitian matrix of rank r can be expressed in the form

$$A_{ik} = MX_i'X_k \qquad (i, k = 1, 2, \cdots, n),$$

where X_i' is the conjugate to X_i and M is real.

Any skew symmetric determinant of odd order is necessarily zero. Hence if a skew symmetric matrix is of odd rank r, all the principal minors of order r vanish. By an argument similar to that above it follows that for an element X_k different from zero the only value of M satisfying the relation $A_{kk} = MX_k^2$ is zero. Hence all the minors A_{ik} vanish. In case the rank r is even the minors have again the form (1).

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A SUBSTITUTE FOR DUHAMEL'S THEOREM.

By Gilbert Ames Bliss.

Every teacher of the calculus, at some period in his course, finds himself facing the difficulties of deducing in an elementary way the expressions for geometrical or physical or mechanical quantities as definite integrals. The methods which he may use must be simple enough, in their rougher and more intuitive forms, to be comprehensible to beginners in the study of the subject, and yet they should be capable of refinement and precision without serious alteration, by filling in interstices, in order to satisfy the rigorous temperament of the more mature mathematician.

Two papers, by W. F. Osgood and R. L. Moore, explaining the use of Duhamel's theorem in this connection, have appeared in the *Annals*,* and have suggested to me the presentation of the present paper. I must confess that the theorem of Duhamel seems to me to be a somewhat clumsy and unsatisfactory instrument, though this is a subject upon which I find wide variations of opinion among my colleagues in various sections of the country. In its original form, and as it is at present stated in many of the elementary text books, the theorem conceals questions of uniform continuity, which are as essential in some form or other to the study of definite integrals as the notions of limits are to the differential calculus, and gives no intimation that something is lacking in the argument. This objection has been removed in the two papers referred to, but in a way which I find difficult to make intelligible to elementary students. For this reason I venture to lay before the readers of the *Annals* the theorem and methods of application explained in the following paragraphs. Needless to say I do not attempt to explain in an elementary course the notion of uniform continuity, notwithstanding its very great importance. But I try to leave a framework of proof in the mind of the student, of such a character that the moment he discovers uniform continuity, that moment he may also understand its application in the deduction of the theorem.

The theorem of § 1, which is the kernel of the paper, is in its essentials similar to those of Osgood and Moore, though different in form.† I do not hope to convince my readers at once of its utility as compared with that

* Osgood, The integral as the limit of a sum and a theorem of Duhamel's, vol. 4 (1903), page 161; Moore, On Duhamel's theorem, vol. 13 (1912), page 161.

† The theorem and some of its applications were incorporated at my suggestion in a master's thesis by Mr. L. B. White entitled "Some applications of uniform continuity in the theory of definite integrals," The University of Chicago, 1910.

of Duhamel's theorem, though in my own courses it has seemed to be effective. But at least I should like to submit it for their consideration.

1. A Theorem Concerning Definite Integral Sums. Let us consider a function of the form $f(p, p', p'')$, where p is a symbol for a set of values (x, y, z), and p' and p'' for analogous sets. The points p, p', p'' are to range over a closed measurable region V in xyz-space in which f is continuous, and hence also uniformly continuous.

If the region V is divided into measurable sub-regions with maximum diameters less than δ and with volumes denoted by ΔV_k $(k = 1, 2, \cdots, n)$, and if in each region three points p_k, p_k', p_k'' are chosen, then

$$(1)\quad \operatorname*{Lim}_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k', p_k'')\Delta V_k = \operatorname*{Lim}_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k, p_k)\Delta V_k = \int_V f(p, p, p,)dV.$$

The proof of the theorem is very simple. Since f is uniformly continuous in V there corresponds to any positive constant ϵ a second positive constant δ such that

$$| f(p_k, p_k', p_k'') - f(p_k, p_k, p_k) | < \epsilon$$

whenever p_k, p_k', p_k'' lie in a sub-region ΔV_k with maximum diameter less than δ. Hence when V is divided into sub-regions with maximum diameters less than δ, as indicated above, the two sums whose limits are to be proved equal differ by a quantity less in absolute value than

$$\epsilon \sum_{k=1}^{n} \Delta V_k = \epsilon V,$$

provided that we let V denote the measure of the original region. The second limit in the formula (1) is by definition the definite integral indicated on the right, and so the theorem is proved.

The statement and proof of the theorem would apply exactly as they stand in case V were a region in one or two or any other number of dimensions; and it is clear that the method applies also if the number of points p selected in ΔV_k were more or less than three.

2. Applications of the Theorem. In the applications of the theorem of the preceding section two principles from the theory of functions of real variables are of great service; first, in any closed region a continuous function has a maximum and a minimum which it attains at points of the region; and second, if the region is connected, every value between the maximum and the minimum is assumed by the function at one point at least. These are statements which I find that younger students accept readily as being intuitively true; in fact, they usually regard one as being somewhat over particular if he dwells at length upon the necessity of proof. It is certain, however, that no one can penetrate far into a thorough understanding of

the calculus without having convinced himself in some way or other of these properties of continuous functions.

The difficulties of the situation are not only mathematical ones, as Osgood has very clearly pointed out in his paper. It is necessary in many of the applications to presuppose properties of physical magnitudes which can not always conveniently be set down in advance. I shall therefore base a first application of the theorem as far as possible upon the properties of attractive force as stated by Osgood in his paper (page 175). The problem is to express as a definite integral the component parallel to the x-axis of the attraction of an arbitrary solid upon a particle of mass m situated exterior to the solid. It was considered by both Osgood and Moore in the papers referred to above.

The following notations are those of Osgood:

m = the mass of the fixed particle;
V = the attracting solid;
ρ = the density of V at a point $p(x, y, z)$;
α = the angle between the line mp and the x-axis.

The density ρ is supposed to be a continuous function of p.

Let V be divided into solids ΔV_k as in § 1 each of which fills out a closed measurable connected portion of space. For the portion ΔV_k the mass will be less than ΔV_k times the maximum of ρ in ΔV_k, greater than ΔV_k times the minimum of ρ, and hence equal to $\rho_k \Delta V_k$, where ρ_k is an intermediate value between the maximum and minimum and is surely taken on by ρ at some point p_k in ΔV_k. The attraction of ΔV_k for the particle m will similarly be expressible as $\kappa m \rho_k \Delta V_k / r_k'^2$, where κ is a gravitational constant and r_k' is an intermediate value between the maximum and the minimum of r in ΔV_k. The projection of the attraction on the x-axis is expressible in the form $\kappa m \rho_k \cos \alpha_k'' \Delta V_k / r_k'^2$, where $\cos \alpha_k''$ is intermediate between the maximum and the minimum of $\cos \alpha$ in ΔV_k and furthermore is a value which $\cos \alpha$ takes at some point p_k'' of ΔV_k. Hence the total x-component of the attraction of V for m is expressible in the form

$$X = \sum_{k=1}^{n} \kappa m \rho_k \cos \alpha_k'' \Delta V_k / r_k'^2 . \tag{2}$$

This would end our labors if we knew how to compute conveniently the coördinates of p_k, p_k', p_k''. But as we do not, it is necessary to take the limit of the sum on the right as δ approaches zero. According to the theorem of § 1 the value of this limit is also

$$X = \lim_{\delta=0} \sum_{k=1}^{n} \kappa m \rho_k \cos \alpha_k \Delta V_k / r_k^2 = \int_V \frac{\kappa m \rho \cos \alpha}{r^2} dV$$

found by changing p_k' and p_k'' both to p_k in the sum (2), α_k and r_k denoting the values of α and r at the point p_k.

The area of a surface generated by revolving a plane curve about the x-axis may be defined, in the well-known way, to be the limit of the sum of the lateral areas of certain inscribed cones. The sum of these lateral areas has the form

$$2\pi \sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{1 + [f'(x_k')]^2}\Delta x_k,$$

where the equation of the curve is $y = f(x)$ and x_k' s a suitably chosen value in the interval Δx_k of which x_{k-1} and x_k are the ends. According to the theorem of § 1, for regions of one dimension, the limit of this sum has the value

$$\lim_{\Delta x=0} 2\pi \sum_{k=1}^{n} f(x_k)\sqrt{1 + [f'(x_k)]^2}\Delta x_k = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx.$$

The pressure of a liquid upon the plane wall of a containing vessel can be computed in a similar way. We admit (1) that on a horizontal area A

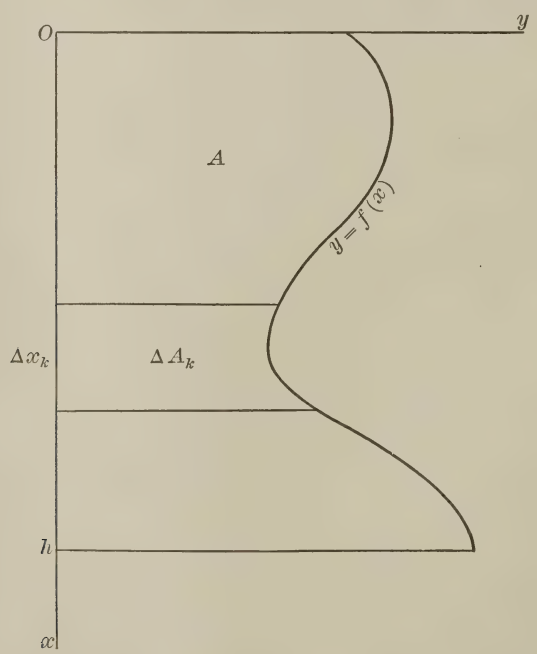


Fig. 1.

at a depth x in a liquid there is a pressure equal to wxA, where w is the weight of a cubic unit of the liquid; (2) that upon any area B whatsoever there is a pressure greater than or equal to the pressure upon an equal horizontal

area at the depth of the highest point of B, and less than or equal to the pressure upon an equal horizontal area at the depth of the lowest point of B; and (3) that the pressure upon the sum of a number of areas is the sum of the pressures on the areas.

Let the area on which the pressure is to be computed be of the form shown in the accompanying figure, the y-axis being in the surface of the liquid. The function $f(x)$ is supposed to be continuous and the interval 0–h is divided into n parts Δx_k $(k = 1, 2, \cdots, n)$. The area ΔA_k has the value

$$\Delta A_k = f(x_k)\Delta x_k,$$

where $f(x_k)$ is a suitably chosen ordinate between the smallest and the largest ordinates of the interval Δx_k. The pressure on ΔA_k has the value

$$\Delta P_k = w x_k' f(x_k)\Delta x_k,$$

where x_k' is a suitably chosen abscissa between the two which bound the interval Δx_k and define the highest and lowest points of ΔA_k. Hence the total pressure on the area A is

$$P = \sum_{k=1}^{n} w x_k' f(x_k)\Delta x_k.$$

As the intervals Δx_k are decreased in size the expression on the right is always equal to the pressure P, and, by the theorem of § 1 for a region of one dimension and with two points p_k and p_k' only instead of three, we have

$$P = \lim_{\Delta x=0} \sum_{k=1}^{n} w x_k f(x_k)\Delta x_k = w \int_0^h x f(x)\,dx.$$

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THE POINTS OF INFLEXION OF A PLANE CUBIC CURVE.

By L. E. Dickson.

CONTENTS.

§ 1. Introduction. § 2. Homogeneous coördinates. § 3. Euler's theorem; singular points. § 4. The Hessian covariant. § 5. Inflexion points of a cubic curve. § 6. Inflexion triangles. § 7. Usual canonical form of a ternary cubic. § 8. Nature of the coefficients of the resolvent quartic. § 9. The equation X for the abscissas of the nine inflexion points. § 10. Galois group G a linear group. § 11. Structure of the linear group L. § 12. Equation X solvable by radicals. § 13. Group of the resolvent quartic. § 14. Group G for a general cubic curve. § 15. Determination of the inflexion points.

1. Introduction. The object of the first half of this paper is to give a self-contained and elementary exposition of the geometrical side of the theory of the inflexion points of a cubic curve without singular points. The object of the second half is to present the algebraic side of the theory, including proofs that the equation X of the ninth degree to which the problem leads is solvable by radicals, a determination of the Galois group of X for certain special cubic curves and for the general one, and a proof that X can be solved by means of a quartic and two cubic equations and hence by means of three cube roots and four square roots, no one of which can be dispensed with in general.

This interesting problem affords an excellent illustration of the complete mastery over an intricate algebraic situation which is possible by the use of Galois' theory of algebraic equations. Readers having little or no acquaintance with that theory will be able to see from this illuminating concrete example what the theory really means and what it can accomplish.

In his Traité des substitutions, Jordan laid the foundations for the applications of Galois' theory to problems of geometry and analysis, and in particular proved (p. 303) that the group of our equation X is a subgroup of the linear group L.

In his Algebra, 2d ed., vol. 2, pp. 390–418, Weber treated the problem from the geometrical and group standpoints and made (p. 416–7) some correct but unproved statements as to the actual group of equation X.

While the earlier results in the present paper are classic, the methods employed are largely novel and the exposition is especially elementary. The material in §§ 11–15 relating to the definitive determination of the Galois group is believed to be new.

2. Homogeneous Coördinates. Let x and y be the Cartesian coördi-

nates of a point referred to rectangular axes and let

$$a_i x + b_i y + c_i = 0 \qquad (i = 1, 2, 3)$$

be the equations of three straight lines L_i $(i = 1, 2, 3)$ forming a triangle. Then

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0.$$

Choose the sign before the radical so that

$$p_i = \frac{a_i x + b_i y + c_i}{\pm\sqrt{a_i^2 + b_i^2}}$$

is positive for a point (x, y) inside the triangle $L_1L_2L_3$; then p_i is the length of the perpendicular from that point to L_i. The homogeneous cöordinates of a point (x, y) are three numbers x_1, x_2, x_3 such that

$$\rho x_1 = k_1 p_1, \quad \rho x_2 = k_2 p_2, \quad \rho x_3 = k_3 p_3,$$

where k_1, k_2, k_3 are constants, the same for all points, while ρ is arbitrary. Thus only the ratios of x_1, x_2, x_3 are defined. The coefficients of $k_i p_i$, which are proportional to a_i, b_i, c_i, will henceforth be denoted by the latter letters. After this slight change of notation, we have

$$\rho x_i = a_i x + b_i y + c_i, \quad \Delta \neq 0 \qquad (i = 1, 2, 3). \tag{1}$$

Solving equations (1) by determinants, we get

$$\Delta x = \rho \Sigma A_i x_i, \quad \Delta y = \rho \Sigma B_i x_i, \quad \Delta = \rho \Sigma C_i x_i,$$

where A_i, B_i, C_i denote the cofactors of a_i, b_i, c_i in Δ. Hence

$$x = \frac{A_1 x_1 + A_2 x_2 + A_3 x_3}{C_1 x_1 + C_2 x_2 + C_3 x_3}, \qquad y = \frac{B_1 x_1 + B_2 x_2 + B_3 x_3}{C_1 x_1 + C_2 x_2 + C_3 x_3}. \tag{2}$$

Given an equation $f(x, y) = 0$ of degree n in Cartesian coördinates, we derive by means of (2) a homogeneous equation $\phi(x_1, x_2, x_3)$ of degree n in homogeneous coördinates. In particular, any straight line is represented by an equation of the first degree in x_1, x_2, x_3, and conversely. For example, $x_1 = 0$ represents a side of the triangle of reference.

Let y_1, y_2, y_3 be the homogeneous coördinates of the same point (x, y) referred to a new triangle of reference $L_1'L_2'L_3'$. As before

$$\rho y_i = a_i' x + b_i' y + c_i' \qquad (i = 1, 2, 3), \tag{1'}$$

where the right member equated to zero represents L_i'. Solving these as

before, we obtain x and y as linear fractional functions of y_1, y_2, y_3. Inserting these values into (1), we get formulas like

$$x_i = c_{i1}y_1 + c_{i2}y_2 + c_{i3}y_3, \quad D = |c_{ij}| \neq 0 \quad (i = 1, 2, 3). \tag{3}$$

Thus a change of triangle of reference gives rise to a linear homogeneous transformation of the homogeneous coördinates.

3. Euler's Theorem; Singular Points. Let $f(x_1, x_2, x_3)$ be a homogeneous polynomial of degree n, and let

$$t = kx_1^a x_2^b c_3^c \qquad (a + b + c = n)$$

be any term of f. Then

$$x_1\frac{\partial t}{\partial x_1} = at, \qquad x_2\frac{\partial t}{\partial x_2} = bt, \qquad x_3\frac{\partial t}{\partial x_3} = ct,$$

and their sum is nt. Hence we have Euler's theorem

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} = nf. \tag{4}$$

If there is a set of solutions x_1, x_2, x_3, not all zero, of

$$\frac{\partial f}{\partial x_1} = 0, \qquad \frac{\partial f}{\partial x_2} = 0, \qquad \frac{\partial f}{\partial x_3} = 0,$$

and hence of $f = 0$, by (4), the point (x_1, x_2, x_3) is called a singular point of the curve $f = 0$. It is essential to know that the definition of a singular point is independent of the particular triangle of reference used. Under the transformation (3) of coördinates, let f become $\phi(y_1, y_2, y_3)$, so that $\phi = 0$ represents our same curve, but referred to the new triangle of reference. Then if (x_1, x_2, x_3) is a singular point of $f = 0$, the same point (y_1, y_2, y_3) is a singular point of $\phi = 0$ since

$$\frac{\partial \phi}{\partial y_j} = \sum_{i=1}^{3}\frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial y_j} = 0 \qquad (j = 1, 2, 3).$$

It is easily proved that two or more branches of the curve cross at a singular point. But we do not presuppose this geometrical interpretation, since we shall always employ the above analytic test.

4. The Hessian Covariant. The Hessian of f is

$$h = \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{vmatrix}.$$

Let transformation (3), of determinant D, replace f by $\phi(y_1, y_2, y_3)$. The product hD is a determinant of the third order in which the element in the ith row and jth column is the sum of the products of the elements of the ith row of h by the corresponding elements of the jth column of D, and hence is

$$\frac{\partial^2 f}{\partial x_i \partial x_1} c_{1j} + \frac{\partial^2 f}{\partial x_i \partial x_2} c_{2j} + \frac{\partial^2 f}{\partial x_i \partial x_3} c_{3j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_j} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_j} + \frac{\partial f}{\partial x_3}\frac{\partial x_3}{\partial y_j}\right)$$
$$= \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial y_j}\right).$$

Let D' be the determinant obtained from D by interchanging its rows and columns. By the same rule of multiplication of determinants, the element in the rth row and jth column of the determinant equal to $D' \cdot hD$ is

$$c_{1r}\frac{\partial}{\partial x_1}\left(\frac{\partial \phi}{\partial y_j}\right) + c_{2r}\frac{\partial}{\partial x_2}\left(\frac{\partial \phi}{\partial y_j}\right) + c_{3r}\frac{\partial}{\partial x_3}\left(\frac{\partial \phi}{\partial y_j}\right) = \frac{\partial}{\partial y_r}\left(\frac{\partial \phi}{\partial y_j}\right),$$

since c_{ir} is the partial derivative of x_i, given by (3), with respect to y_r. Hence

$$D^2 h = \left|\frac{\partial^2 \phi}{\partial y_r \partial y_j}\right|_{r,\ j=1,\ 2,\ 3} = \text{Hessian of } \phi.$$

Thus any linear transformation of determinant D which replaces f by ϕ also replaces $D^2 h$ by H, where h is the Hessian of f and H is the Hessian of ϕ. In other words, $H = 0$ represents the same curve as $h = 0$ when referred to the new triangle of reference. Accordingly, in investigating the relations between a curve $f = 0$ and its Hessian curve $h = 0$, we may without loss of generality assume that $f = 0$ is referred to any special triangle of reference.

5. Inflexion Points of a Cubic Curve. Let $f(x_1, x_2, x_3)$ be a homogeneous polynomial of the third degree, such that $f = 0$ has no singular point. Choose a triangle of reference having the vertex $P = (0, 0, 1)$ on the cubic curve $f = 0$. Then there is no term involving x_3^3. In the terms $rx_1x_3^2$ and $sx_2x_3^2$, r and s are not both zero, since otherwise P would be a singular point. Taking $rx_1 + sx_2$ as the new variable x_1, we see that f becomes

$$x_3^2 x_1 + x_3(ax_1^2 + bx_1x_2 + cx_2^2) + \phi(x_1, x_2).$$

Replacing x_3 by $x_3 - (ax_1 + bx_2)/2$, we get

$$f_1 = x_3^2 x_1 + ex_3x_2^2 + C,$$

where C is a cubic function of x_1, x_2, whose second derivative with respect

to x_i and x_j shall be designated by C_{ij}. The Hessian of f_1 is

$$h_1 = \begin{vmatrix} C_{11} & C_{12} & 2x_3 \\ C_{12} & C_{22} + 2ex_3 & 2ex_2 \\ 2x_3 & 2ex_2 & 2x_1 \end{vmatrix} = -8ex_3^3 + \cdots.$$

Thus $P = (0, 0, 1)$ is on $h_1 = 0$ if and only if $e = 0$.

Let d be the coefficient of x_2^3 in C. Then $x_1 = 0$ meets $f_1 = 0$ in the points for which $(ex_3 + dx_2)x_2^2 = 0$. Two of these points are identical with P, so that $x_1 = 0$ is tangent to $f_1 = 0$ at P. The three* points coincide if and only if $e = 0$, and P is then called an inflexion point of $f_1 = 0$ and $x_1 = 0$ the inflexion tangent at P. Hence P is on $h_1 = 0$ if and only if it is an inflexion point of $f_1 = 0$. In view of the remark at the end of § 4, we have

THEOREM 1. *Each intersection of a cubic curve $f = 0$ without a singular point with its Hessian curve $h = 0$ is an inflexion point of $f = 0$, and conversely.*

There is at least one intersection of $f = 0$, $h = 0$. For, by eliminating x_3, we get a homogeneous equation in x_1 and x_2, having therefore at least one set of solutions x_1', x_2'. Then for $x_1 = x_1'$, $x_2 = x_2'$, our equations $f = 0$, $h = 0$ have at least one common root $x_3 = x_3'$. Then (x_1', x_2', x_3') is an intersection and therefore a point of inflexion of $f = 0$.

Take this point as the vertex $(0, 0, 1)$ of a triangle of reference. As above, we get $f_1 = 0$, where now $e = 0$, $d \neq 0$. Taking a suitable multiple of x_2 as a new x_2, we have $d = 1$. We add a suitable multiple of x_1 to x_2 to delete the term $x_2^2 x_1$, and get

$$F = x_3^2 x_1 + C, \qquad C = x_2^3 + 3bx_2x_1^2 + ax_1^3. \tag{5}$$

Its Hessian H is the determinant h_1 for $e = 0$. Thus

$$H = 2x_1 \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} - 4x_3^2 C_{22}$$

$$= 72x_1(bx_2^2 + ax_1x_2 - b^2x_1^2) - 24x_3^2x_2.$$

Eliminating x_3^2 between $F = 0$, $H = 0$, we get

$$x_2^4 + 6bx_2^2x_1^2 + 4ax_2x_1^3 - 3b^2x_1^4 = 0.$$

* If $d = e = 0$, f_1 has the factor x_1. But if $f_1 = x_1Q$,

$$\frac{\partial f_1}{\partial x_1} = Q + x_1\frac{\partial Q}{\partial x_1}, \qquad \frac{\partial f_1}{\partial x_2} = x_1\frac{\partial Q}{\partial x_2}, \qquad \frac{\partial f_1}{\partial x_3} = x_1\frac{\partial Q}{\partial x_3}$$

all vanish at a point of intersection of $x_1 = 0$, $Q = 0$, whereas $f_1 = 0$ has no singular point.

If $x_1 = 0$, then $x_2 = 0$ and we get the known inflexion point (0, 0, 1). For the remaining intersections, we may set $x_1 = 1$. Then for each root of

$$r^4 + 6br^2 + 4ar - 3b^2 = 0, \tag{6}$$

we get two inflexion points $(1, r, \pm s)$, where, by $F = 0$,

$$-s^2 = r^3 + 3br + a. \tag{7}$$

In fact, no root of (6) makes $s = 0$; in other words, (6) has no double root. For, by eliminating r between the quartic and cubic equations, we get $a^2 + 4b^3 = 0$. But, the three derivatives of F with respect to x_1, x_2, x_3 are all zero at $(1, x, 0)$ if

$$x^2 + b = 0, \qquad 2bx + a = 0.$$

These are satisfied by $x = 0$ if $b = 0$, and by $x = -a/(2b)$ if $b \neq 0$, whereas $F = 0$ has no singular point. Hence

$$a^2 + 4b^3 \neq 0 \tag{8}$$

and we have proved

THEOREM 2. *Any plane cubic curve without a singular point has exactly nine distinct inflexion points.*

6. Inflexion Triangles. For a fixed root r of (6), the three points $P = (0, 0, 1)$, $(1, r, \pm s)$ are collinear, being on the line $x_2 = rx_1$. We may start with any one of the nine inflexion points in place of P. Hence they lie by threes on $9 \cdot 4/3$ lines.

THEOREM 3. *The straight line joining any two inflexion points of a cubic curve without singular points meets the curve in a new inflexion point. The nine inflexion points lie by threes upon twelve straight lines, four of which pass through any one of the nine points.*

The six inflexion points not on a particular one of these lines lie by threes upon two further lines, and the three lines are said to form an inflexion triangle. There are 12/3 such triangles.

THEOREM 4. *The nine inflexion points lie by threes upon the sides of any one of the four inflexion triangles.*

The last theorem follows also from the important formula

$$\frac{1}{24}H + rF = (rx_1 - x_2)\left\{x_3^2 - \frac{1}{r}(rx_2 + kx_1)^2\right\}, \tag{9}$$

where $k = (r^2 + 3b)/2$. The two straight lines represented by the last factor contain the six inflexion points $(1, \rho, \pm\sigma)$, where ρ is a root $\neq r$ of (6) and, corresponding to (7),

$$-\sigma^2 = \rho^3 + 3b\rho + a.$$

The condition is $r\sigma^2 = (r\rho + k)^2$, which follows from (6) and

$$\rho^3 + r\rho^2 + (6b + r^2)\rho + r^3 + 6br + 4a = 0,$$

the equation satisfied by the roots $\mp r$ of (6).

Hence, for each root r of (6), $H + 24rF = 0$ is an inflexion triangle. A like result follows at once for our initially given cubic curve $f = 0$. For, f was transformed into F by a linear transformation of a certain determinant δ, and F is replaced by a like form by the transformation which multiplies x_3 by δ and x_1 by δ^{-2}, of determinant δ^{-1}. Hence the product of the two transformations is of determinant unity and replaces f by a form F. Hence it replaces the Hessian h of f by the Hessian H of F (§ 4). If the reduction of f to F were actually carried out, a and b would be found as certain functions of the coefficients of f. Indirect methods of finding a and b are given later.

THEOREM 5. *If $f = 0$ is any cubic curve without singular points and if h is the Hessian of f, then, for each root r of* (6), *$h + 24rf = 0$ is an inflexion triangle of $f = 0$.*

7. Usual Canonical Form of a Ternary Cubic. Consider a plane cubic curve C_3 without a singular point. Take as triangle of reference one having two vertices (1, 0, 0) and (0, 1, 0) at two inflexion points and two sides $x_2 = 0$, $x_1 = 0$ as the corresponding inflexion tangents. Let $f = 0$ be the equation of C_3 referred to this triangle. Then

$$f \equiv x_2\phi + tx_3^3 \equiv x_1\psi + tx_3^3,$$

where t is a constant, while ϕ and ψ are quadratic functions. Thus

$$f = x_1x_2l + tx_3^3, \qquad l \equiv a_1x_1 + a_2x_2 + a_3x_3. \tag{10}$$

Then $a_1a_2 \neq 0$. For, if $a_1 = 0$, the three partial derivatives of f vanish when $x_1 = 1$, $x_2 = x_3 = 0$ and (1, 0, 0) would be a singular point. Similarly, if $a_2 = 0$, (0, 1, 0) would be a singular point.

Evidently $l = 0$ is an inflexion tangent; it meets $x_3 = 0$ at an inflexion point distinct from (1, 0, 0) and (0, 1, 0), since $a_1a_2 \neq 0$. We thus have another proof of the first part of Theorem 3.

Take as new variables z_1, z_2, x_3, where

$$a_1x_1 = \omega z_1 + \omega^2 z_2 + c, \qquad a_2x_2 = \omega^2 z_1 + \omega z_2 + c, \qquad c = -\tfrac{1}{3}a_3x_3,$$

where ω is an imaginary cube root of unity. Then

$$-l = z_1 + z_2 + c,$$

$$-a_1a_2x_1x_2l = \begin{vmatrix} c & z_1 & z_2 \\ z_2 & c & z_1 \\ z_1 & z_2 & c \end{vmatrix} = z_1^3 + z_2^3 + c^3 - 3z_1z_2c,$$

$$-a_1a_2f = z_1^3 + z_2^3 - (a_1a_2t + \tfrac{1}{27}a_3^3)x_3^3 + a_3z_1z_2x_3.$$

The coefficient of x_3^3 is not zero, since otherwise the three partial derivatives of f would vanish for $z_1 = z_2 = 0$ and there would be a singular point. Hence by taking a suitable multiple of x_3 as z_3, we get

$$f = \alpha(z_1^3 + z_2^3 + z_3^3) + 6\beta z_1 z_2 z_3. \tag{11}$$

We saw that $x_3 = 0$ intersects the cubic (10) in three inflexion points. Hence $z_3 = 0$ intersects (11) in three inflexion points. By symmetry, the same is true of $z_1 = 0$ and of $z_2 = 0$. Since a non-vanishing coördinate may be taken to be unity, we obtain at once the nine points of inflexion of (11):

$$\begin{array}{lll} (0, & 1, & -1) \quad (0, \quad 1, \quad -\omega) \quad (0, \quad 1, \quad -\omega^2) \\ (-1, & 0, & 1) \quad (-\omega, \quad 0, \quad 1) \quad (-\omega^2, \quad 0, \quad 1) \\ (1, & -1, & 0) \quad (1, \quad -\omega, \quad 0) \quad (1, \quad -\omega^2, \quad 0). \end{array} \tag{12}$$

The Hessian of (11) is $6^3 g$, where

$$g = -\alpha\beta^2(z_1^3 + z_2^3 + z_3^3) + (\alpha^3 + 2\beta^3) z_1 z_2 z_3. \tag{13}$$

The functions f and g are linearly independent since

$$d \equiv \alpha(\alpha^3 + 8\beta^3) \neq 0. \tag{14}$$

In fact, the partial derivatives of f vanish if

$$\alpha z_1^2 + 2\beta z_2 z_3 = \alpha z_2^2 + 2\beta z_1 z_3 = \alpha z_3^2 + 2\beta z_1 z_2 = 0,$$

which are satisfied when $z_1 = z_2 = \beta$, $z_3 = -\alpha/2$, if $\alpha^3 + 8\beta^3 = 0$, and when $z_1 = z_2 = 1$, $z_3 = 0$ if $\alpha = 0$. In view of (14), the intersections of $f = 0$, $g = 0$ are those of $f = 0$, $z_1 z_2 z_3 = 0$. Hence the nine points (12) are the only inflexion points of $f = 0$.

The three points in the ith row of table (12) lie on the line $z_i = 0$ in view of their origin. The three in the first, second and third columns lie in the respective lines

$$z_1 + z_2 + z_3 = 0, \qquad \omega^2 z_1 + \omega z_2 + z_3, \qquad \omega z_1 + \omega^2 z_2 + z_3 = 0. \tag{15}$$

If we multiply the first coördinate of each of the points in the respective columns (12) by ω, we obtain three sets of three points, the first set being those in the main diagonal of table (12) and the other two sets being those in the positions of the elements of a determinant of the third order which yield the remaining two positive terms. Hence they lie by threes upon the lines derived from (15) by replacing z_1 by $\omega^2 z_1$.

But if we multiply the first coördinate of each point in the respective columns by ω^2, we get those corresponding to negative terms of the determinant, which therefore lie by threes upon the lines derived from (15) by replacing z_1 by ωz_1.

We thus have a new proof of the second part of Theorem 3 as well as of Theorem 4. Moreover, we now see just how the nine inflexion points are distributed into twelve sets of three collinear points. This distribution can be described most briefly by employing for the nine inflexion points the notation $[\xi\eta]$, where ξ and η take the values 0, 1, 2 independently. To the table (12) we make correspond

$$\begin{array}{lll} [00] & [12] & [21] \\ [11] & [20] & [02] \\ [22] & [01] & [10]. \end{array} \tag{16}$$

THEOREM 6. *Three inflexion points* $[\xi_1, \eta_1]$, $[\xi_2, \eta_2]$, $[\xi_3, \eta_3]$ *are collinear if and only if*

$$\xi_1 + \xi_2 + \xi_3 \equiv 0, \qquad \eta_1 + \eta_2 + \eta_3 \equiv 0 \pmod 3. \tag{17}$$

From (11) and (13), we get

$$dz_1z_2z_3 = \alpha g + \alpha\beta^2 f, \qquad d\Sigma z_1{}^3 = -6\beta g + (\alpha^3 + 2\beta^3)f. \tag{18}$$

By the first, $g + \beta^2 f = 0$ gives one inflexion triangle. The product of the left members of the sides (15) of another inflexion triangle is

$$\begin{vmatrix} z_1 & z_2 & z_3 \\ z_3 & z_1 & z_2 \\ z_2 & z_3 & z_1 \end{vmatrix} = \Sigma z_1{}^3 - 3z_1z_2z_3 = \frac{(\alpha + 2\beta)}{d}[-3g + (\alpha - \beta)^2 f],$$

by (18). The remaining two inflexion triangles are obtained from the last by replacing z_1 by $\omega^2 z_1$ or ωz_1. In view of (18), we have only to replace α by $\omega^2\alpha$ or $\omega\alpha$ in our final result. We thus get the

LEMMA. *The four inflexion triangles of* (11) *are* $3g + rf = 0$, *where*

$$r = 3\beta^2, \quad -(\alpha - \beta)^2, \quad -(\omega\alpha - \beta)^2, \quad -(\omega^2\alpha - \beta)^2. \tag{19}$$

The quartic having these four numbers as roots is easily found. First, α, $\omega\alpha$, $\omega^2\alpha$ are the roots of $x^3 - \alpha^3 = 0$. Diminishing each root by β, we obtain the equation

$$(y + \beta)^3 - \alpha^3 = 0.$$

Then $(\alpha - \beta)^2$, etc., are the roots of a cubic obtained by transposing the terms of even degree, squaring, and replacing y^2 by z. By inspection, we get

$$z(z + 3\beta^2)^2 = (-3\beta z + \alpha^3 - \beta^3)^2.$$

Hence the last three numbers (19) are the roots of the cubic obtained by replacing $-z$ by r. Multiply the new cubic by $r - 3\beta^2$. We get the

desired quartic (6), in which

$$b = \beta(\alpha^3 - \beta^3), \qquad a = \tfrac{1}{4}\alpha^6 - 5\alpha^3\beta^3 - 2\beta^6. \tag{20}$$

We have therefore a second proof of Theorem 5.

8. Nature of the Coefficients of Quartic (6). For two special cubic curves (5) and (11), we have determined the coefficients a and b of (6). It is important for the sequel to know that, just as in these two cases, a and b are always rational functions of the coefficients of f.

First, the four roots of (6) are the only values of r for which $\phi \equiv h + 24rf$ has a linear factor l. For, $l = 0$ meets $f = 0$ in three points on $h = 0$, which are therefore inflexion points. Thus l is one of the twelve lines through three inflexion points, and there are two companion lines l_1 and l_2 such that $ll_1l_2 = h + 24tf$, where t is a root of (6). By hypothesis, $lQ = \phi$. By subtraction, we see that, if $r \neq t$, f has the factors l and $Q - l_1l_2$ and hence has a singular point (foot-note in § 5).

Next, if $x_1 - dx_2 - ex_3$ is a linear factor of ϕ and we replace x_1 by $dx_2 + ex_3$ in ϕ, we obtain a cubic function of x_2, x_3, whose four coefficients must vanish. Eliminating d and e, we obtain two conditions in which r and the coefficients of f enter rationally and integrally. If we free the greatest common divisor of their left members from multiple factors, we must obtain a quartic function of r whose coefficients are rational in those of f.

THEOREM 7. *The four inflexion triangles of a cubic curve $f = 0$ without singular points depend upon a quartic* (6) *whose coefficients a and b are rational* functions of the coefficients of f.*

9. The Equation X for the Abscissas of the Nine Inflexion Points. Let R denote the set or domain of all rational functions with rational coefficients of the coefficients of the equation $f = 0$ of the cubic curve. After applying a linear transformation on x_1, x_2, x_3, with coefficients in R, we may assume that $x_3 \neq 0$ for each inflexion point (we have only to take a new triangle of reference whose side $x_3 = 0$ does not pass through a point of inflexion). Pass from homogeneous to Cartesian coördinates by setting $x = x_1/x_3$, $y = x_2/x_3$. After rotating the axes through a suitable angle with a rational sine and cosine, we may assume that the y-axis is not parallel to any line joining two inflexion points. Then the abscissas $x_1, \cdots, x_9$ of the inflexion points are distinct, and the equations (with coefficients in R) of the curve and its Hessian curve uniquely determine the ordinate y_i of the inflexion point (x_i, y_i) as a rational function of x_i with coefficients in R. Thus, by eliminating y^3 and y^2 between the equations

* In fact, integral functions, since otherwise a root r would be infinite for certain f's. The expressions for $S = b$, $T = 4a$ are invariants of f, first computed by Aronhold and given in full in Salmon's Higher Plane Curves, §§ 221–2. No use is made of these long invariants in this paper.

of these two curves, we obtain y expressed as a rational function of x in R with denominator not zero at an inflexion point. By substituting this expression for y into $f = 0$, we obtain the equation X for the nine abscissas of the inflexion points. The relation

$$\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix} = 0$$

between the coördinates of three collinear inflexion points gives a rational relation $\psi(x_i, x_j, x_k) = 0$, with coefficients in R, between their abscissas. By means of $\psi = 0$ and X, we can express x_i as a rational function of x_j and x_k with coefficients in R (Theorem 3). Consider, conversely, three inflexion points whose abscissas satisfy $\psi = 0$ and let the line joining two of them (x_j, y_j) and (x_k, y_k) meet the curve at (x, y); the latter is an inflexion point, so that $x = x_i$ and hence $y = y_i$.

THEOREM 8. *Three inflexion points are collinear if and only if their abscissas satisfy the rational relation $\psi = 0$ with coefficients in R.*

Let G be the group of permutations of $x_1, \cdots, x_9$ such that every rational function of $x_1, \cdots, x_9$ with coefficients in R which is unaltered by all of the permutations of G equals a number in the domain R, and conversely. Then G is called the Galois group of equation X for the domain R.

Let a permutation of G replace three roots x_1, x_2, x_3 for which $\psi(x_1, x_2, x_3) = 0$ by the roots x_r, x_s, x_t. By the converse property just stated, $\psi(x_r, x_s, x_t) = 0$. Hence Theorem 8 yields

THEOREM 9. *Every permutation of the Galois group G of the equation X for the nine abscissas of the inflexion points replaces the abscissas of any three collinear inflexion points by the abscissas of three collinear inflexion points.*

10. Galois Group G a Linear Group. Henceforth, we shall denote the abscissas by the nine symbols $[\xi\eta]$ used in (16) for the corresponding inflexion points. By Theorem 9, G is a subgroup of the group L of those permutations on the nine roots $[\xi\eta]$ which replace three roots forming a row, column, positive or negative term of the determinant of table (16) by a set of three such roots, or, by Theorem 6, which replace three roots satisfying congruences (17) by three roots also satisfying them.

For each set of integers $a, \cdots, C$, the linear substitution

$$(21) \quad \xi' \equiv a\xi + b\eta + c, \qquad \eta' \equiv A\xi + B\eta + C, \qquad \begin{vmatrix} a & b \\ A & B \end{vmatrix} \not\equiv 0 \pmod 3$$

gives rise to a permutation of the nine roots $[\xi\eta]$. For, it replaces two distinct roots $[\xi_1\eta_1]$ and $[\xi_2\eta_2]$ by distinct roots $[\xi_1'\eta_1']$ and $[\xi_2'\eta_2']$. If the

latter were identical, then

$$a(\xi_1 - \xi_2) + b(\eta_1 - \eta_2) \equiv 0, \quad A(\xi_1 - \xi_2) + B(\eta_1 - \eta_2) \equiv 0 \pmod 3,$$

and, in view of the determinant of (21), $\xi_1 - \xi_2 \equiv \eta_1 - \eta_2 \equiv 0$, contrary to hypothesis. Moreover, the resulting permutation belongs to the group L. For, if $[\xi_i \eta_i]$ where $i = 1, 2, 3$, are distinct roots satisfying congruences (17), then $[\xi_i' \eta_i']$ satisfy (17), since

$$\sum_{i=1}^{3} \xi_i' = a \sum_{i=1}^{3} \xi_i + b \sum_{i=1}^{3} \eta_i + 3c \equiv 0, \qquad \sum_{i=1}^{3} \eta_i' \equiv 0 \pmod 3.$$

Conversely, every permutation P of L is induced by a linear substitution (21) on the indices. For, if P replaces [00] by $[cC]$ then $P = P_1 S$, where P_1 is a permutation of L which leaves [00] unaltered and S is that induced by

$$\xi' \equiv \xi + c, \qquad \eta' \equiv \eta + C \pmod 3. \tag{22}$$

Let P_1 replace [10] by $[aA]$, so that a and A are not both zero. Then we can find two integers b and B such that $aB - bA$ is not divisible by 3 (if $a = 1$ or 2, take $b = 0$, $B = 1$; if $A = 1$ or 2, take $b = 1$, $B = 0$). Hence $P_1 = P_2 S_1$, where S_1 is the permutation induced by

$$\xi' \equiv a\xi + b\eta, \qquad \eta' \equiv A\xi + B\eta, \qquad aB - bA \not\equiv 0 \pmod 3, \tag{23}$$

while P_2 is the permutation of L which alters neither [00] nor [10] and hence not [20] in view of the diagonal term of table (16). Let P_2 replace [01] by $[de]$, so that $e \not\equiv 0$. Then $P_2 = P' S_2$, where S_2 is the permutation induced by

$$\xi' \equiv \xi + d\eta, \qquad \eta' \equiv e\eta,$$

while P' is a permutation of L which leaves fixed [00], [10], [20] and [01], and hence [22] by the third row of (16), then [11] and [12] by the first and second columns, and finally [21] and [02] by the first and second rows. Thus P' is the identity and $P = P_1 S = P_2 S_1 S = S_2 S_1 S$. Since the product of three linear substitutions (21) is such a substitution, P is induced by a linear substitution.

Now $[cC]$ was any one of 3×3 roots, $[aA]$ any one of $3^2 - 1$ roots, $[de]$ any one of 3×2 roots. We thus have

THEOREM 10. *The Galois group G is a subgroup of the group L of the $9 \cdot 8 \cdot 6$ linear substitutions* (21) *on the indices ξ, η.*

11. Structure of the Linear Group L. There are exactly $3^2 - 1$ incongruent linear homogeneous functions of ξ and η with integral coefficients modulo 3, viz.,

$$\pm \xi, \qquad \pm \eta, \qquad \pm (\xi + \eta), \qquad \pm (\xi - \eta).$$

Hence they are permuted by the linear homogeneous substitutions (23), which form a group H. The same is true of their squares $\xi^2, \cdots$. The identity I and

J: $\qquad \xi' \equiv -\xi, \qquad \eta' \equiv -\eta$

are the only substitutions (23) which leave unaltered each of the four squares. For, if the sign of ξ is changed, while η is not changed, or vice versa, then $(\xi + \eta)^2$ becomes $(\xi - \eta)^2$. Hence there is a (2, 1) correspondence between the 48 substitutions of H and the permutations on the four squares, so that H corresponds to the symmetric group G_{24} on four letters. As well known, G_{24} has an invariant (self-conjugate) subgroup G_{12}, which has an invariant subgroup G_4, with in turn an invariant G_2. It follows that H has a series of subgroups of orders 24, 8, 4, 2, each invariant in the preceding, so that H is a solvable group.

The group T of the nine "translations" (22) is invariant in L. In fact, (23) transforms (22) into the translation

$$\xi' \equiv \xi + ac + bC, \qquad \eta' \equiv \eta + Ac + BC \pmod 3.$$

Since T has an invariant subgroup of order 3, we have

THEOREM 11. *The linear group L is a solvable group.*

12. Equation X Solvable by Radicals. If we make use of the well known theorems that any subgroup of a solvable group is a solvable group and that an equation is solvable by radicals if its group is a solvable group, we obtain, from Theorems 10 and 11,

THEOREM 12. *If R is the domain defined by the coefficients of the equation $f = 0$ of a cubic curve without singular points, and if X is the equation for the abscissas of its nine inflexion points, the Galois group G of X for R is a solvable group, and equation X is solvable by radicals.*

But we need not presuppose the theorem that any subgroup of a solvable group is solvable in order to conclude that X is solvable by radicals. For, we shall prove that G is identical with the solvable group L in case the coefficients of f are independent variables. Since therefore X is then solvable by radicals, it will continue to be solvable by radicals when one or more of the variable coefficients of f have been given special values. However, the use of the theorem that the subgroups of a solvable group are solvable enables us to derive the important Theorem 12 without entering upon the difficult question of the actual determination of the Galois group G. Similarly,* we may draw important conclusions as to the number of real bitangents to a quartic curve or real lines on a cubic surface, if we have merely proved that the group G of the corresponding equation is a subgroup of a certain group and have not actually determined G.

* Dickson, *Annals of Mathematics*, vol. 6 (1905), p. 141.

13. Group of the Resolvent Quartic (6). If $r_1, \cdots, r_4$ are the roots of (6), it is shown in texts on the theory of equations* that

$$y_1 = r_1r_2 + r_3r_4, \qquad y_2 = r_1r_3 + r_2r_4, \qquad y_3 = r_1r_4 + r_2r_3$$

are the roots of

$$y^3 - 6by^2 + 12b^2y - 16a^2 - 72b^3 = 0.$$

Setting $y = z + 2b$, we obtain the reduced cubic

$$z^3 = D \equiv 16(a^2 + 4b^3),$$

where $D \neq 0$ by (8). The discriminant of (6) is known to equal the discriminant of $z^3 - D$ and hence is $-27D^2$.

Consider the case $a = 1$, $b = -1$. Then $D = -48$ is not the cube of a rational number, so that each y_i is irrational. Hence, in view of Ferrari's method of solving quartic equations, our present form of (6)

$$r^4 - 6r^2 + 4r - 3 = 0 \tag{6'}$$

is not the product of two quadratic factors with rational coefficients. Nor has it a rational root, since no root equals ± 1, ± 3. Hence (6′) is irreducible in the domain $R(1)$ of rational numbers. The group of (6′) for $R(1)$ is therefore transitive; it is not a subgroup of the G_8 which leaves y_i unaltered; nor is it the alternating group G_{12} since the discriminant of (6′) was seen to be $-27D^2$ and hence not the square of a rational number. Hence the group of (6′) for $R(1)$ is the symmetric group G_{24}. Since a 24 valued function of the roots of (6′) is therefore a root of an irreducible equation of degree 24, the same is true of the quartic (6) when the coefficients of f are arbitrary variables.

THEOREM 13. *Let the coefficients of the equation $f = 0$ of the cubic curve be independent variables and let R be the domain defined by them. Then the group of* (6) *for R is the symmetric group G_{24}.*

COROLLARY. *The adjunction of the four roots of* (6) *to R reduces the group G of equation X for R to an invariant subgroup Σ of index* 24 *under G.*

The Corollary follows from Theorem 13 and the Theorem of Jordan,† since the adjunction of all nine roots of X reduces the group of (6) to the identity. For, the abscissas and hence the ordinates (§ 9) of the inflexion points are in the enlarged domain. Thus the ratios of the coefficients of the equation of the line through collinear inflexion points are in that domain. Since three such lines form an inflexion triangle, it follows from Theorem 5 that each root r of (6) is in the enlarged domain.

* Cf. Dickson's Elementary Theory of Equations, 1914, p. 39.

† Traité des substitutions, p. 269; cf. Dickson's Introduction to the Theory of Algebraic Equations, p. 81.

14. Group G for a General Cubic Curve. Adjoin the root of (6) associated with the inflexion triangle whose sides are determined by the sets of inflexion points

$$[00],\ [01],\ [02];\ [10],\ [11],\ [12];\ [20],\ [21],\ [22], \tag{24}$$

corresponding to the negative terms of the expansion of the determinant of (16). Then G reduces to the subgroup which permutes these three sets (24). This is true of the substitution which merely changes the sign of ξ, or that which merely changes the sign of η, or of

$$\xi' \equiv \xi + c, \qquad \eta' \equiv A\xi + \eta + C \pmod 3,$$

which replaces the kth set (24) by the $(k + c)$th set (24). These three types evidently generate the group of the $4 \cdot 3^3$ substitutions (21) with $b \equiv 0$, $aB \not\equiv 0$, A, c, C arbitrary modulo 3, whose index under the total group L is 4. Since there are four inflexion triangles, we now have all the substitutions of L which permute the sets (24).

We now adjoin also the root of (6) which is associated with the inflexion triangle determined by the three sets of inflexion points

$$[00],\ [10],\ [20];\ [01],\ [11],\ [21];\ [02],\ [12],\ [22],$$

corresponding to the positive terms of the expansion of the determinant of (16). Since these sets are derived from the sets (24) by the interchange of ξ and η, the largest subgroup of L which permutes these sets is the group of substitutions (21) with $A \equiv 0$. After both adjunctions, the group is contained in that composed of

$$\xi' \equiv a\xi + c, \qquad \eta' \equiv B\eta + C, \qquad aB \not\equiv 0 \pmod 3. \tag{25}$$

Of these, the substitutions

$$\xi' \equiv a\xi + c, \qquad \eta' \equiv a\eta + C \qquad (a = 1 \text{ or } -1) \tag{26}$$

permute the rows (16) and also the columns, while $\xi' \equiv \xi$, $\eta' \equiv -\eta$ interchanges the rows and columns. The latter is therefore true of (25) if $a \equiv -B$. We have now proved

THEOREM 14. *After the adjunction of all four roots of* (6), *the group G of X reduces to a subgroup Σ of the group of the* 18 *substitutions* (26).

Information concerning Σ is gained from special cubic curves.

First, we take $b = 0$, $a = -2$ in (5). Then (6) becomes $r(r^3 - 8) = 0$. Hence after the adjunction of the four roots, the domain is $R(\omega)$, where ω is an imaginary cube root of unity. But the sides of any one of the inflexion triangles involve the radical $\sqrt{2}$, which is not in $R(\omega) = R(\sqrt{-3})$. For $r = 0$, the triangle is given by the Hessian $-24x_2(x_3^2 + 6x_1^2)$ of F. For

$r \neq 0$, the triangle is given by (9), two of whose sides involve the radical $\sqrt{r}$ and hence the new radical $\sqrt{2}$. Thus Σ is here of order 2, while the group G of X for the domain of the rational numbers is of order 4.

Next, we employ the special cubic curve

$$f = x_1^3 + 2x_2^3 + 4x_3^3 + 6x_1x_2x_3. \tag{27}$$

It is reduced to the canonical form (11) with $\alpha = 2$, $\beta = 1$, by

$$x_1 = \sqrt[3]{2}\, z_1, \qquad x_2 = z_2, \qquad x_3 = z_3/\sqrt[3]{2},$$

a transformation of determinant unity. The roots (19) of (6) are in the domain $R(\omega)$. By (15) and the remarks following it, the sides of the four inflexion triangles involve the single new irrationality $\sqrt[3]{2}$. Hence Σ is here of order 3, while G is of order 6.

Consider the cubic curve the coefficients of whose equation are independent variables and let R be the domain defined by them. In view of the last results, the order of the group Σ is divisible by 2 and 3. Since (26) is of period 2 if $a = -1$, a substitution (26) of period 3 has $a = +1$. Interchanging the variables if necessary, we may assume that ξ is altered. Then the substitution or its square is $\xi' \equiv \xi + 1$, $\eta' \equiv \eta + t$. Introducing ξ and $\eta - t\xi$ as new variables, we obtain a group Σ_1 conjugate with Σ under L and having the substitution $\xi' \equiv \xi + 1$, $\eta' \equiv \eta$. The only substitutions (21) commutative with this one are those with $a \equiv 1$, $A \equiv 0$, $B \not\equiv 0$, and b, c, C arbitrary, $2 \cdot 3^3$ in number. It is transformed into its inverse by $\xi' = -\xi$, $\eta' = \eta$. Hence exactly $4 \cdot 3^3$ substitutions of L transform into itself the cyclic group of order 3 generated by our substitution. If Σ were of order 6, it would contain a single cyclic group of order 3, which is invariant under G, since Σ is. By the corollary in § 13, G would then be of order 24×6, which exceeds $4 \cdot 3^3$. From this contradiction with the preceding result, we conclude that Σ contains all 18 substitutions (26). The order of G thus equals that of H.

THEOREM 15. *If the coefficients of a cubic curve $f = 0$ are independent variables, the group of the equation X for the abscissas of the nine inflexion points $[\xi\eta]$, $\xi, \eta = 0, 1, 2$, for the domain of the coefficients of f, is the group L of all linear substitutions on ξ and η.*

15. Determination of the Inflexion Points. Let the coefficients of the equation of the cubic curve be independent variables. The solution of quartic (6) requires the extraction of a cube root and three square roots, since its group is G_{24}. After the adjunction of its four roots, the group of X is that of the 18 substitutions (26). The product $h + 24rf$ of the linear functions which vanish at the sides of an inflexion triangle (Theorem 5) has as coefficients numbers in the enlarged domain. The determination of the

linear factors requires the solution of a cubic equation. For example, for (27) and $r = -1$, the factors are $x_1 + \lambda x_2 + 2\lambda^{-1}x_3$, where $\lambda^3 = 2$. Consider the inflexion triangle determined by (24); after the adjunction of the roots of the corresponding cubic equation, the group permutes the roots in each set (24). The only substitutions (26) having this property are $\xi' \equiv \xi$, $\eta' \equiv \eta + C$, which form a group G_3. The group of this resolvent cubic is therefore of order 6. In the new domain, the group of the corresponding resolvent cubic for another inflexion triangle is G_3, whose solution requires only the extraction of a cube root. After the adjunction of the latter, we have the sides of two inflexion triangles, and their intersections give the nine inflexion points.

THEOREM 16. *The inflexion points can be found by solving a quartic and two cubic equations, and the solution involves three cube roots and four square roots. For a general cubic curve, no one of these radicals can be avoided or expressed in terms of the others.*

THE UNIVERSITY OF CHICAGO,
May, 1914.

PROPERTIES OF FOUR CONFOCAL PARABOLAS WHOSE VERTICAL TANGENTS FORM A SQUARE.

By C. M. Hebbert.

In a recent paper* I have shown that if the diagonals of a quadrilateral are perpendicular and equal, an infinite number of squares may be circumscribed to the quadrilateral, and that if two squares can be inscribed in a given quadrilateral, an infinite number of squares may be inscribed in it. In an article in the Bulletin de l'Academie Imperiale des Sciences de St. Petersbourg, vol. VII, p. 177, Thomas Clausen pointed out an apparent, though only apparent, dualism existing between the squares in these two cases, but was unable to give a satisfactory explanation. This paper shows that the squares inscribed in the quadrilateral are a single infinity of the ∞^2 squares circumscribed to an infinite set of S-quadrangles, i. e., quadrangles whose diagonals are perpendicular and equal.

The equations of four confocal parabolas, which we call G-parabolas, whose vertical tangents form a square, may be written in the form

$$(G_1) \quad y^2 = 4px + 4p^2, \qquad (G_2) \quad y^2 = 4(p - c)x + 4(p - c)^2,$$

$$(G_3) \quad x^2 = 4q^2 - 4qy, \qquad (G_4) \quad x^2 = 4(c - q)y + 4(c - q)^2,$$

where c is the side of the square formed by the vertical tangents and the common focus is at the origin F (see figure).

The equations of tangents to G_1 and G_2 having the same slope m are

$$(t_1) \quad y = mx + \frac{m^2 + 1}{m} p, \qquad \text{and} \qquad (t_2) \quad y = mx + \frac{m^2 + 1}{m} (p - c).$$

The equations of tangents to G_3 and G_4 which are perpendicular to t_1 and t_2 are

$$(t_3) \quad y = -\frac{1}{m}x + q\frac{m^2 + 1}{m^2}, \qquad \text{and} \qquad (t_4) \quad y = -\frac{1}{m}x + \frac{m^2 + 1}{m^2}(q - c).$$

Inspection of equations (t_1) and (t_2) shows that the distance between them is equal to the distance between the lines (t_3) and (t_4). Hence we have

* "The inscribed and circumscribed squares of a quadrilateral and their significance in kinematic geometry," *Annals of Math.*, Vol. 16, p. 38. In the present paper this is referred to as "the preceding paper." By an unfortunate oversight references to the work of Thomas Clausen, referred to in this paper, and Carnot, Geometrie de Position, p. 374, were omitted in that article.

THEOREM I. *Any four rectangular tangents to this system of parabolas form a square.*

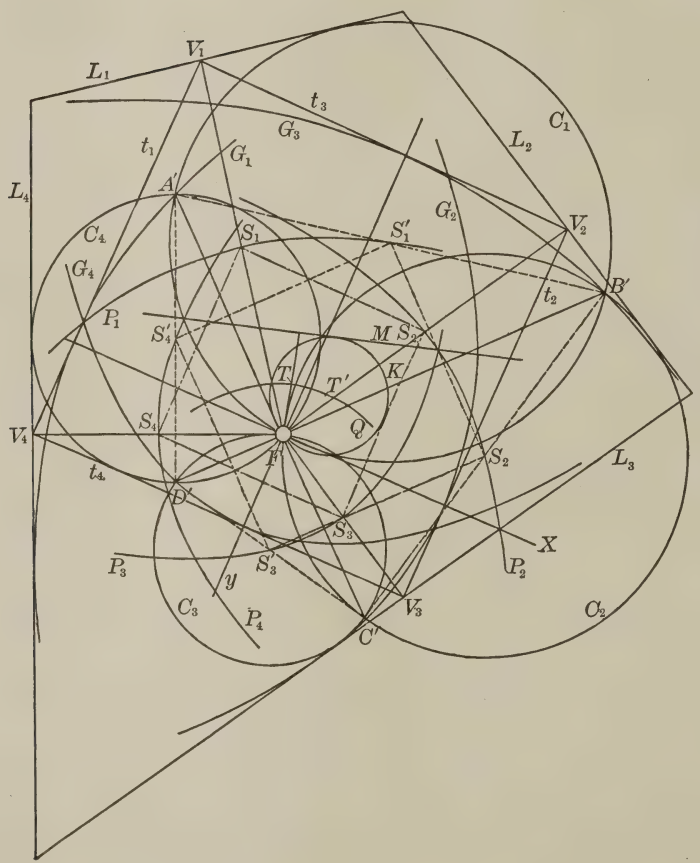
The sect-point of t_1 and t_3 is the point $(q/m - p,\ q + p/m)$, which lies on the line $p(p + x) = q(y - q)$, which is designated by L_1.

Loci of sect-points (t_2, t_3), (t_2, t_4), (t_1, t_4), respectively, are

$$(L_2) \qquad (p - c)(x + p - c) = q(y - q),$$

$$(L_3) \qquad (p - c)(x + p - c) = (q - c)(y - (q - c)),$$

$$(L_4) \qquad p(x + p) = (q - c)(y - (q - c)).$$



These are tangents to G_1, G_2, G_3, G_4, in pairs, i. e., each of the lines L_1, L_2, L_3, L_4, is tangent to two of the parabolas and each of the parabolas is tangent to two of the lines.

The coördinates of the center of the square formed by the four tangents

are

$$\left(\frac{2q + (m-1)c}{2m} - p, \qquad q - \frac{(m+1)c - 2p}{2m}\right);$$

and the locus of the center is, therefore, the line

$$(M) \qquad (2y - (2q - c))(2q - c) = (2p - c)(2x + 2p - c).$$

THEOREM II. *The vertices of the variable square formed by the rectangular tangents of Theorem I always lie on straight lines which are tangent to the parabolas in pairs. The center of the variable square lies on a fixed straight line.*

The results of this paper thus far, if considered in connection with the results of the preceding paper, show that there is a (1, 1) correspondence between points of the parabolas and points of the sides of the quadrilateral whose inscribed squares envelope these parabolas. For, in the preceding paper we saw that the quadrilateral uniquely determined the four confocal parabolas while in this section we have shown that the four confocal parabolas uniquely determine a quadrilateral. To every point on a side of the quadrilateral corresponds a side of an inscribed square and thence a point of one of the parabolas, and conversely. We shall use this property instead of carrying out some extended analytic processes in what follows.

Any line, $y = nx$, through the focus, cuts the parabolas G_1 and G_2 in the points

$$A': \qquad \left(\frac{2p \pm 2p\sqrt{1+n^2}}{n^2}, \qquad \frac{2p \pm 2p\sqrt{1+n^2}}{n}\right),$$

and

$$C': \quad \left(\frac{2(p-c) \pm 2(p-c)\sqrt{1+n^2}}{n^2}, \qquad \frac{2(p-c) \pm 2(p-c)\sqrt{1+n^2}}{n}\right),$$

respectively.

The length of the segment cut out of the line by these two points is $d_1 = 2c(1 \pm \sqrt{1+n^2})\sqrt{1+n^2}/n^2$.

The line $y = -x/n$, perpendicular to $y = nx$, cuts G_3 and G_4 in the points B' $(2q \pm 2q\sqrt{1+n^2})/n$, $(-2q \pm 2q\sqrt{1+n^2})/n^2)$ and D' $((2(q-c) \pm 2(q-c)\sqrt{1+n^2})/n$, $(-2(q-c) \pm 2(q-c)\sqrt{1+n^2})/n^2)$ respectively, and the length of their join is $d_2 = d_1$, above.

From this and Theorem IV, Corollary 1, of the preceding paper follows

THEOREM III. *Any two perpendicular lines through the focus of this system of parabolas cut the two opposite pairs of this system in four points forming an S-quadrangle, i. e., a quadrangle about which may be circumscribed an infinite number of squares.*

Since the coördinates of these four points in the same order satisfy the

four equations t_1, t_2, t_3, t_4, simultaneously, they must be the points of contact of a set of rectangular tangents and n is a function of m: $n(1 - m^2) = 2m$.

Now for a given value of n, the vertices of the infinite set of squares belonging to the S-quadrangle so determined lie on four fixed circles, C_1, C_2, C_3, C_4. These four circles must be tangent to the four lines, L_1, L_2, L_3, L_4, on which the vertices of the squares circumscribed to the four G-parabolas lie. For, if the circles cut the lines, there would be two squares having the same points of contact with the G-parabolas and having their vertices on the four fixed lines. This is impossible since there is a (1, 1) correspondence between the points of these four lines and the four parabolas.

The same reasoning applies to the locus of centers of squares corresponding to given values of n, i. e., the circle K which is the locus of the centers of squares passing through these four points is tangent to the line M on which lie the centers of all the squares tangent to the system of parabolas. Analytically, the proof is as follows:

The circle, K, is determined by the focus F and the mid-points of the segments cut out of $y = nx$ and $y = -x/n$ by the parabolas, i. e., the mid-points of $A'C'$ and $B'D'$. The equation of this circle K is

$$x^2 - \frac{(2p + 2nq - c - cn)(1 \pm \sqrt{1 + n^2})}{n^2} x + y^2 - \frac{(c - 2q + 2pn - cn)(1 \pm \sqrt{1 + n^2})}{n^2} y = 0.$$

The equation of M, which we have determined, may be written

$$y = \frac{(4p - 2c)x + (2p - c)^2 + (2q - c)^2}{4q - 2c}.$$

The sect-point of M and K, determined by substituting this value of y in the equation of K and reducing, has an abscissa given by the value of x in the equation

$$4n^2x^2 + 4n[n(2p - c) - (1 \pm \sqrt{1 + n^2})(2q - c)]x + [n(2p - c) - (1 \pm \sqrt{1 + n^2})(2q - c)]^2 = 0.$$

Since this expression is a perfect square and M is not parallel to the y-axis, M is tangent to the circle, K.

Hence we have five infinite sets of circles C_1, C_2, C_3, C_4, K tangent to five fixed lines, L_1, L_2, L_3, L_4, M. Since the circles, whatever the value of n, pass through the focus F and are always tangent to these five fixed lines, their centers describe five parabolas, P_1, P_2, P_3, P_4, Q whose vertices are at the mid-points of the perpendiculars from the focus to the five fixed lines. The feet of four of these perpendiculars are the vertices of the square

formed by the four vertical tangents to the fixed parabolas and the center of this square is at the opposite extremity of the diameter of the circle K which has one extremity at the focus. Moreover, this center is on the line M and therefore, the line M is perpendicular to the diameter mentioned, i. e., the center of the square formed by the vertical tangents is at the foot of the perpendicular from the focus F to the line M. Hence the vertex of the parabola Q, which is the mid-point of this perpendicular, is the center of the square formed by the vertices of the four P-parabolas.

This would follow directly from the fact that the S-quadrangle in this case is formed by the four vertices of the G-parabolas and the vertices of the P-parabolas and the Q-parabola are the centers of the circles belonging to this S-quadrangle.

The results of this paper may be summed up as follows:

Given four confocal parabolas whose vertical tangents form a square;

THEOREM I. *Any four rectangular tangents to these four parabolas form a square whose vertices lie on four fixed lines and whose center lies on a fifth fixed line.*

THEOREM II. *Any two perpendicular lines through the common focus cut the four parabolas in four points which form an S-quadrangle, about which an infinite number of squares may be circumscribed whose vertices lie on four circles tangent to the four fixed lines of Theorem I, and whose centers lie on a fifth circle tangent to the fifth fixed line of Theorem I.*

THEOREM III. *The centers of the four circles in Theorem II lie on four parabolas confocal with the given parabolas, whose vertices are the mid-points of the sides of the quadrilateral formed by the vertices of the given parabolas, and therefore form a square. The center of this square is the vertex of a parabola, confocal with the others of the set, which is the locus of the center of the fifth circle in Theorem II.*

Hence we have nine confocal parabolas, five fixed lines, and five infinite sets of circles tangent to the five fixed lines and concurrent at the common focus of the parabolas. These seem to explain the apparent (though only apparent) dualism of the theorems relating to quadrilaterals having an infinite number of inscribed squares and quadrangles having an infinite number of circumscribed squares.*

* For mention of this apparent dualism and doubt as to its nature, see the article by Thomas Clausen, l. c.

SOME REMARKS ON CONFORMAL REPRESENTATION.*

By T. H. Gronwall.

1. In recent investigations of the uniformization of analytic functions, the following proposition† is of fundamental importance:

When the analytic function

$$z = f(x) = \sum_{n=1}^{\infty} a_n x^n \tag{1}$$

effects the conformal representation of the circle $|x| < 1$ *on a simple region*‡ *in the z-plane, the area of this region being less than or equal to A, then, for every* $r < 1$, *the expressions* $|f(x)|$ *and* $|f'(x)|$ *have upper boundaries for* $|x| \leqq r$ *which depend on r and A only, and not on the coefficients in* (1).

Koebe gives the inequality

$$|f(x)| < \sqrt{A}\left(\frac{r}{\pi(1-r)^2} + 1\right) \qquad \text{for} \qquad |x| \leqq r < 1$$

and Courant the corresponding inequality for $f'(x)$

$$|f'(x)| \leqq \sqrt{\frac{A}{\pi}} \cdot \frac{1}{1-r^2}.$$

We shall now show that *for* $|x| \leqq r < 1$

$$\begin{aligned} |f(x)| &\leqq \sqrt{\frac{A}{\pi}} \cdot \sqrt{\log \frac{1}{1-r^2}}, \\ |f'(x)| &\leqq \sqrt{\frac{A}{\pi}} \cdot \frac{1}{1-r^2}, \end{aligned} \tag{2}$$

and that these upper boundaries cannot be replaced by any smaller ones.

Under the assumption made on $f(x)$, the circle $|x| < r$ is conformally represented on a simple region in the z-plane, the area of which is

$$A(r) = \int_0^r d\rho \int_0^{2\pi} |f'(\rho e^{\vartheta i})|^2 \rho d\vartheta = \pi \sum_{n=1}^{\infty} n |a_n|^2 r^{2n}, \tag{3}$$

and since this region is interior to that corresponding to the circle $|x| < 1$,

* Read before the American Mathematical Society, October 31, 1914.

† Koebe, Journal für Mathematik, 138 (1910), p. 222. Courant, Math. Annalen, 71 (1912), p. 164.

‡ Simple region = schlichter Bereich, a simply connected and nowhere overlapping region.

we have

$$\pi \sum_{n=1}^{\infty} n \mid a_n \mid^2 r^{2n} < A$$

for any $r < 1$. A fortiori, the inequality

$$\pi \sum_{n=1}^{N} n \mid a_n \mid^2 r^{2n} < A,$$

subsists, and letting r tend toward unity, we obtain

$$\pi \sum_{n=1}^{N} n \mid a_n \mid^2 \leqq A,$$

whence, observing that the terms of the series are positive, we find by increasing N indefinitely

$$\pi \sum_{n=1}^{\infty} n \mid a_n \mid^2 \leqq A. \tag{4}$$

On the other hand, it is evident that for $\mid x \mid \leqq r$

$$\mid f(x) \mid \leqq \sum_{n=1}^{\infty} \mid a_n \mid r^n. \tag{5}$$

In the Lagrange inequality

$$\left(\sum_{n=1}^{\infty} u_n v_n \right)^2 \leqq \sum_{n=1}^{\infty} u_n{}^2 \cdot \sum_{n=1}^{\infty} v_n{}^2,$$

where u_n, v_n are any positive quantities such that the series are convergent, we now make $u_n = \sqrt{n} \mid a_n \mid$, $v_n = r^n / \sqrt{n}$ and obtain from (5),

$$\mid f(x) \mid^2 \leqq \sum_{n=1}^{\infty} n \mid a_n \mid^2 \cdot \sum_{n=1}^{\infty} \frac{r^{2n}}{n} = \sum_{n=1}^{\infty} n \mid a_n \mid^2 \cdot \log \frac{1}{1 - r^2},$$

whence, using (4), we immediately obtain the first inequality (2). To show that the upper boundary obtained is the smallest possible, consider the special function

$$f_0(x) = \sqrt{\frac{A}{\pi}} \cdot \frac{\log \dfrac{1}{1 - r_0 x}}{\sqrt{\log \dfrac{1}{1 - r_0{}^2}}},$$

where $0 < r_0 < 1$. This function effects the conformal representation of the unit circle on a simple region, since for any two different values of x inside the unit circle the values of $f_0(x)$ are different, and from (3) it is seen at once that the area of this region equals A. Furthermore, for $x = r_0$, we have

$$\mid f_0(r_0) \mid = \sqrt{\frac{A}{\pi}} \cdot \sqrt{\log \frac{1}{1 - r_0{}^2}},$$

so that the upper boundary (2) is actually reached for the particular value $r = r_0$.

To prove the second inequality (2), we observe that for $|x| \leqq r$

$$|f'(x)| \leqq \sum_{n=1}^{\infty} n |a_n| r^{n-1};$$

applying Lagrange's inequality with $u_n = \sqrt{n}\,|a_n|$, $v_n = \sqrt{n}\,r^{n-1}$, we find

$$|f'(x)|^2 \leqq \sum_{n=1}^{\infty} n |a_n|^2 \cdot \sum_{n=1}^{\infty} n r^{2n-2} = \sum_{n=1}^{\infty} n |a_n|^2 \cdot \frac{1}{(1-r^2)^2},$$

and combining with (4), the desired result is obtained. Using the comparison function

$$f_0(x) = (1 - r_0^2)\sqrt{\frac{A}{\pi}} \cdot \frac{x}{1 - r_0 x}$$

it is seen as before that the upper boundary cannot be replaced by any smaller one.

2. In connection with his exposition of Koebe's distortion theorem, Fricke* proves the following proposition:

When

$$f(x) = \frac{1}{x} + \sum_{n=1}^{\infty} a_n x^n, \tag{6}$$

where the power series has no constant term and converges for $|x| < 1$, *and* $f(x)$ *does not take the same value for two different values of* x *inside the unit circle, then the maximum* $M(r)$ *of* $|f(x)|$ *for* $|x| = r$ *satisfies the inequality*

$$M(r) < \frac{2 + 2\sqrt{2}}{r}, \qquad 0 < r < 1.$$

We shall now show that *this inequality may be replaced by*

$$M(r) < \frac{9}{4} \cdot \frac{1}{r}. \tag{7}$$

Since $f(x)$ takes each of its values but once in the unit circle, $z = f(x)$ effects the conformal representation of the unit circle on a simple region in the z-plane, containing the point $z = \infty$ (corresponding to $x = 0$) in its interior; to the circumference $|x| = r$ $(0 < r < 1)$ there corresponds a closed curve C_r, all the points of which are at finite distance, and if $r > r_0$, the curve C_r lies entirely inside the curve C_{r_0}. Consequently

* R. Fricke and F. Klein, Vorlesungen über die Theorie der automorphen Functionen, vol. 2 (Leipzig, Teubner, 1912), pp. 497–498.

$M(r) < M(r_0)$, and

$$rM(r) < rM(r_0) < M(r_0), \qquad 0 < r_0 < r < 1.$$

But since $xf(x)$ is holomorphic for $| x | < 1$, we have for $r \leqq r_0$

$$rM(r) = \max_{|x|=r} | xf(x) | \leqq \max_{|x|=r_0} | xf(x) | = r_0 M(r_0) < M(r_0),$$

and this, together with the preceding inequality, gives

$$rM(r) < M(r_0), \qquad 0 \leqq r < 1. \tag{8}$$

We shall now establish an upper boundary for $M(r_0)$, and then determine r_0 so as to make this boundary as small as possible. The area inside the curve C_r is readily found to be

$$\pi\left(1 - \sum_{n=1}^{\infty} n \, | a_n |^2 r^{2n}\right),$$

and this area being obviously positive for every $r < 1$, we find, letting r tend toward unity in the same way as when proving (4), that

$$\sum_{n=1}^{\infty} n \, | a_n |^2 \leqq 1. \tag{9}$$

Now

$$M(r) \leqq \frac{1}{r} + \sum_{n=1}^{\infty} | a_n | r^n,$$

and applying the Lagrange inequality and the relation (9) to the right hand series in the same way as before, we see that

$$M(r) \leqq \frac{1}{r} + \sqrt{\log \frac{1}{1 - r^2}}.$$

In the interval $0 \leqq r \leqq 1$, the last expression has only one minimum $= 2.242 \cdots$ for $r = 0.74 \cdots$; taking the latter value for r_0, it follows from (8) that

$$rM(r) < 2.242 \cdots < \frac{9}{4} \qquad \text{for} \qquad 0 < r < 1,$$

which proves our statement (6).

The particular function

$$f(x) = \frac{1}{x} + e^{2\alpha i} x, \tag{10}$$

where α is real, effects a conformal representation of the unit circle on that part of the z-plane which lies outside of the rectilinear cut from

$z = -2e^{\alpha i}$ to $z = 2e^{\alpha i}$. For this function we have

$$M(r) = \frac{1}{r} + r \leqq \frac{2}{r} \qquad \text{for} \qquad 0 < r \leqq 1,$$

so that the constant in (7) cannot be replaced by any value less than 2. It is probable that, for every function (6) satisfying the conditions stated, $M(r) \leqq \frac{1}{r} + r$, the equality sign holding only when $f(x)$ has the form (10). The proof of this statement seems very difficult; the case, however, when all a_n are real and not negative, may be disposed of as follows. From the assumptions concerning $f(x)$, it is seen that $f'(x) \neq 0$ inside the unit circle, and consequently

$$f'(r) = -\frac{1}{r^2} + \sum_{n=1}^{\infty} n a_n r^{n-1} < 0 \qquad \text{for} \qquad 0 < r < 1,$$

since otherwise $f'(x) = 0$ would have a positive root $\leqq r$. Consequently, letting r tend toward unity in the same way as before, we find

$$\sum_{n=1}^{\infty} n a_n \leqq 1,$$

or

$$a_1 \leqq 1 - \sum_{n=2}^{\infty} n a_n,$$

whence

$$M(r) = \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leqq \frac{1}{r} + r - \sum_{n=2}^{\infty} a_n (nr - r^n) \leqq \frac{1}{r} + r,$$

the equality sign holding only for $a_1 = 1$ and $a_n = 0$, $(n > 1)$.

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ON THE MAXIMUM MODULUS OF AN ANALYTIC FUNCTION.*

By T. H. Gronwall.

Carathéodory and Fejér† have proved the following theorem: *Among all analytic functions*

$$F(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots$$

which are holomorphic for $|x| \leqq r$ *and have the same initial* $n+1$ *coefficients* $a_0, a_1, \cdots, a_n$, *there exists one,* $F = f(x)$, *such that the maximum modulus of* $f(x)$ *for* $|x| = r$, *which we denote by* $M(a_0, a_1, \cdots, a_n; r)$, *is smaller than that of any other* $F(x)$. *This* $f(x)$ *is uniquely determined by the coefficients* $a_0, a_1, \cdots, a_n$ *and the radius* r; *it is rational, of degree not exceeding* n, *and of constant modulus*

$$|f(x)| = M(a_0, a_1, \cdots, a_n; r)$$

in every point of the circumference $|x| = r$.

The original proof of Carathéodory and Fejér is based on considerations belonging to Minkowski's geometry of convex solids, and E. Fischer‡ has shown how to derive the same result from the algebraic conditions for the definiteness of certain Hermitian forms. On account of the importance of the theorem, the following quite elementary proof may be of some interest.

First consider the case $n = 0$, that is, only a_0 is given. From the well-known relation

$$\sum_{\nu=0}^{\infty} |a_\nu|^2 r^{2\nu} \leqq \max |F(x)|^2 \qquad \text{for} \qquad |x| = r,$$

it follows that $\max |F(x)|$ for $|x| = r$ always exceeds $|a_0|$ unless $F(x) = a_0$; therefore we have $f(x) = a_0$ and $M(a_0; r) = |a_0|$.

Now supposing our theorem proved when $a_0, a_1, \cdots, a_{n-1}$ are given, we may show in the following manner that the theorem is also true when $a_0, a_1, \cdots, a_n$ are given.

* Read before the American Mathematical Society, April 26, 1913.

† "Über den Zusammenhang der Extremen von harmonischen Funktionen mit ihren Koeffizienten und über den Picard-Landauschen Satz," Rendiconti del Circolo Matematico di Palermo, 32 (1911), pp. 218–239.

‡ "Über das Carathéodorysche Problem, Potenzreihen mit positivem reellen Teil betreffend," ibid., pp. 240–256.

Denote by

$$M = M(a_0, a_1, \cdots, a_n; r)$$

the inferior limit of the maximum of $|F(x)|$ for $|x| = r$; then it is possible to select a sequence of functions $F_1(x)$, $F_2(x)$, $\cdots$ with the given initial coefficients, holomorphic for $|x| \leqq r$ and such that, writing

$$M_\nu = \max |F_\nu(x)| \qquad \text{for} \qquad |x| = r,$$

we have

$$\lim_{\nu=\infty} M_\nu = M. \tag{1}$$

The linear substitution

$$\zeta = \frac{z - \alpha}{1 - \bar{\alpha} z}, \qquad z = \frac{\alpha + \zeta}{1 + \bar{\alpha}\zeta} \qquad (|\alpha| < 1, \ \bar{\alpha} \text{ conjugate to } \alpha) \tag{2}$$

defines a conformal representation of the circle $|z| \leqq 1$ on the circle $|\zeta| \leqq 1$ such that $z = \alpha$ and $\zeta = 0$ are corresponding points.

Now write*

$$G_\nu(x) = \frac{\dfrac{F_\nu(x)}{M_\nu} - \dfrac{a_0}{M_\nu}}{1 - \dfrac{\bar{a}_0}{M_\nu}\dfrac{F_\nu(x)}{M}} = C_{\nu,0}x + C_{\nu,1}x^2 + \cdots + C_{\nu,n-1}x^n + \cdots; \tag{3}$$

then, since $|F_\nu(x)/M_\nu| \leqq 1$ for $|x| \leqq r$ and in particular $|a_0/M_\nu| < 1$, it follows from (2) that

$$|G_\nu(x)| \leqq 1 \qquad \text{for} \qquad |x| \leqq r,$$

and consequently, making

$$H_\nu(x) = \frac{G_\nu(x)}{x} = C_{\nu,0} + C_{\nu,1}x + \cdots + C_{\nu,n-1}x^{n-1} + \cdots, \tag{4}$$

we have, by Schwarz's lemma,

$$|H_\nu(x)| \leqq \frac{1}{r} \qquad \text{for} \qquad |x| \leqq r. \tag{5}$$

Since our theorem is true when the initial n coefficients are given, there exists a rational function $h_\nu(x)$ of degree $\leqq n - 1$, holomorphic for $|x| \leqq r$, its expansion beginning with

$$h_\nu(x) = C_{\nu,0} + C_{\nu,1}x + \cdots + C_{\nu,n-1}x^{n-1} + \cdots,$$

* In § 15 of his paper "Über den Picardschen Satz," Vierteljahrschrift der Naturforschenden Gesellschaft in Zürich, 51 (1906), pp. 252–318, Landau considers the converse of our problem, viz., the determination of the greatest r for which $M(a_0, a_1, \cdots, a_n; r)$ will take a given value M. He also uses the linear substitution (2); but his method is entirely different from that of the present paper and does not bring out the constancy of $|f(x)|$ on the circumference $|x| = r$.

of constant modulus m_ν for $|x| = r$, and such that

$$m_\nu \leqq \max |H_\nu(x)| \qquad \text{for} \qquad |x| = r,$$

or by (5)

$$m_\nu \leqq \frac{1}{r}. \tag{6}$$

Being of constant modulus for $|x| = r$, the rational function $h_\nu(x)$ of degree $\leqq n-1$ may be written in the form*

$$h_\nu(x) = m_\nu e^{\gamma_\nu i} \prod_{\lambda=1}^{n-1} \frac{c_{\lambda\nu} - x}{r - \frac{\bar{c}_{\lambda\nu} x}{r}}, \tag{7}$$

where $0 \leqq \gamma_\nu \leqq 2\pi$ and $|c_{\lambda\nu}| \leqq r$.

Since m_ν, γ_ν, $c_{\lambda\nu}$ are all bounded, we may select an infinite sequence of indices $\nu_1^{(1)}, \nu_2^{(1)}, \cdots$ or briefly, a sequence $\{1\}$ such that

$$\lim_{\{1\}} m_\nu = m$$

exists. From the sequence $\{1\}$, we may further select a sequence $\{2\}$ such that

$$\lim_{\{2\}} m_\nu = m, \qquad \lim_{\{2\}} \gamma_\nu = \gamma,$$

and continuing in this way, we arrive at a sequence $\{n+1\}$ such that

$$\lim_{\{n+1\}} m_\nu = m, \qquad \lim_{\{n+1\}} \gamma_\nu = \gamma, \qquad \lim_{\{n+1\}} c_{1\nu} = c_1, \cdots, \qquad \lim_{\{n+1\}} c_{n-1,\nu} = c_{n-1},$$

where obviously

$$m \leqq \frac{1}{r}, \qquad 0 \leqq \gamma \leqq 2\pi, \qquad |c_\lambda| \leqq r \quad (\lambda = 1, 2, \cdots, n+1). \tag{8}$$

We now form the rational function of degree $\leqq n-1$

$$h(x) = me^{\gamma i} \prod_{\lambda=1}^{n-1} \frac{c_\lambda - x}{r - \frac{\bar{c}_\lambda x}{r}} = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + \cdots; \tag{9}$$

then

$$|h(x)| = m \leqq \frac{1}{r} \qquad \text{for} \qquad |x| = r, \tag{10}$$

and since $C_{\nu,0}, C_{\nu,1}, \cdots, C_{\nu,n-1}$ are found from the expansion of (7) as polynomials in $m_\nu e^{\gamma_\nu i}, c_{1\nu}, \cdots, c_{n-1,\nu}, \bar{c}_{1\nu}, \cdots, \bar{c}_{n-1,\nu}$ with coefficients independent of ν, it is obvious that

$$\lim_{\{n+1\}} C_{\nu,0} = C_0, \qquad \lim_{\{n+1\}} C_{\nu,1} = C_1, \cdots, \qquad \lim_{\{n+1\}} C_{\nu,n-1} = C_{n-1}. \tag{11}$$

* See for instance § 3 of T. H. Gronwall, "On analytic functions of constant modulus on a given contour," these *Annals*, 2d series, vol. 14 (1912), pp. 72–80.

Equation (3) shows that $C_{\nu,0}$, $C_{\nu,1}$, $\cdots$, $C_{\nu,n-1}$ are connected with a_0, a_1, $\cdots$, a_n by the relations

$$a_1 = \left(M_\nu - \frac{|a_0|^2}{M_\nu}\right) C_{\nu,0}$$

$$a_2 = \left(M_\nu - \frac{|a_0|^2}{M_\nu}\right) C_{\nu,1} - \bar{a}_0 a_1 C_{\nu,0}$$

.

$$a_n = \left(M_\nu - \frac{|a_0|^2}{M_\nu}\right) C_{\nu,n-1} - \bar{a}_0 a_1 C_{\nu,n-2} - \cdots - \bar{a}_0 a_{n-1} C_{\nu,0}.$$

Since obviously $\lim_{\{n+1\}} M_\nu = M$, it follows from (11) that the relations connecting C_0, C_1, $\cdots$, C_{n-1} with a_0, a_1, $\cdots$, a_n are obtained from the preceding ones by replacing M_ν by M. If we therefore define a function $f(x)$ by the equation

$$g(x) = xh(x) = \frac{\frac{f(x)}{M} - \frac{a_0}{M}}{1 - \frac{\bar{a}_0}{M}\frac{f(x)}{M}} \tag{12}$$

whence

$$f(x) = M \cdot \frac{\frac{a_0}{M} + g(x)}{1 + \frac{\bar{a}_0}{M} g(x)}, \tag{13}$$

it follows that the expansion of $f(x)$ begins with

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots.$$

Furthermore

$$|a_0|^2 + |a_1|^2 r^2 + \cdots + |a_n|^2 r^{2n} \leqq M_\nu, \quad (\nu = 1, 2, \cdots)$$

and letting ν tend toward infinity,

$$|a_0|^2 + |a_1|^2 r^2 + \cdots + |a_n|^2 r^{2n} \leqq M,$$

so that $|a_0| < M$ (unless $a_1 = a_2 = \cdots = a_n = 0$, in which case we have previously found $M = |a_0|$, $f(x) = a_0$); therefore, since by (10) and (12)

$$|g(x)| = |xh(x)| = rm \leqq 1 \text{ for } |x| = r, \tag{14}$$

we find by comparison of (2) and (13) that $f(x)$ is holomorphic for $|x| \leqq r$ and that for $|x| = r$

$$\max |f(x)| \leqq M,$$

where the equality sign holds only if $rm = 1$ in (14). But since M is the inferior limit of the maximum modulus for all functions $F(x)$, it follows

that the equality sign must hold, whence

$$|h(x)| = \frac{1}{r} \text{ for } |x| = r \tag{15}$$

and $|g(x)| = 1$; hence, by (2) and (13)

$$|f(x)| = M \text{ for } |x| = r.$$

We finally have to prove the uniqueness of this function $f(x)$. Consider any function $F_0(x)$ with the initial coefficients $a_0, a_1, \cdots, a_n$, holomorphic for $|x| \leqq r$ and such that $|F_0(x)| \leqq M$ for $|x| = r$, and form $G_0(x)$ and $H_0(x)$ from $F_0(x)$ by equations (3) and (4), M_ν being replaced by M; then $|H_0(x)| \leqq 1/r$ for $|x| = r$, and $H_0(x)$ has the initial coefficients $C_0, C_1, \cdots, C_{n-1}$. Now there exists one and only one function $h_0(x)$ holomorphic for $|x| \leqq r$, having the initial coefficients $C_0, C_1, \cdots, C_{n-1}$, and making $\mu = \max |h_0(x)|$ for $|x| = r$ a minimum. This μ is equal to $1/r$, since otherwise the substitution of $h_0(x)$ for $h(x)$ in (12) would give an $f(x)$ of maximum modulus less than M. Since consequently $|H_0(x)| \leqq \mu$ for $|x| = r$, it follows from the uniqueness of $h_0(x)$ that

$$H_0(x) = h_0(x) = h(x),$$

and therefore

$$F_0(x) = f(x).$$

It is now easy to set up the algebraic equation which defines M in terms of $a_0, a_1, \cdots, a_n$. With an indeterminate M, set up equation (12) and calculate $C_0, C_1, \cdots, C_{n-1}$, which are obviously rational in $M, a_0, \bar{a}_0, a_1, a_2, \cdots, a_n$. Since the maximum modulus of $h(x)$ for $|x| = r$ is $1/r$, we form the equation analogous to (12)

$$xk(x) = \frac{rh(x) - rC_0}{1 - r\bar{C}_0 \cdot rh(x)}$$

and calculate the first $n - 1$ coefficients $c_0, c_1, \cdots, c_{n-2}$ in the expansion of $k(x)$. The maximum of $|k(x)|$ for $|x| = r$ is evidently also $1/r$, and repeating the above process, we finally arrive at an equation

$$xm(x) = \frac{rl(x) - rd_0}{1 - r\bar{d}_0 \cdot rl(x)}$$

where only the first coefficient e_0 in the expansion of $m(x)$ can be obtained in terms of $M, a_0, \bar{a}_0, a_1, \bar{a}_1, \cdots a_n, \bar{a}_n$ and r by means of the recurrent formulas previously obtained. The required equation is then

$$|e_0| = \frac{1}{r},$$

and the value of M is its smallest positive root. It would be of some interest to show algebraically that this equation is identical with the one obtained explicitly in determinant form by Carathéodory and Fejér.

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NOTE ON THE SIMPLE DIFFERENCE EQUATION.

By J. H. M. Wedderburn.

Boole on his Calculus of Finite Differences derives the Euler-Maclaurin sum formula from the expansion of the operator $\Delta^{-1} = (e^D - 1)^{-1}$ in a series of positive powers of D so obtaining the well known result

$$(1)\quad \Delta^{-1}g(x) = \int^x g(x)dx - \tfrac{1}{2}g(x) + \sum_1^\infty (-1)^n \frac{B_{n+1}}{(2n+2)!}\frac{d^{2n+1}g(x)}{dx^{2n+1}} + \omega(x),$$

where $\omega(x)$ is an arbitrary periodic function of period unity. This series is usually divergent and is besides only applicable to functions which possess derivatives of every order.

If instead of expanding $(e^D - 1)^{-1}$ in powers of D we use the series

$$(2)\qquad \frac{1}{e^D - 1} = \frac{1}{D} - \frac{1}{2} + \sum_1^\infty \left[\frac{1}{D + 2n\pi i} + \frac{1}{D - 2n\pi i}\right],$$

we obtain another form of the solution of the equation $\Delta f(x) = g(x)$, namely

$$(3)\quad f(x) = -\tfrac{1}{2}g(x) + \int^x g(t)dt + 2\sum_1^\infty \int^x \cos 2\pi n(x-t)g(t)dt + \omega(x),$$

the lower limits being arbitrary constants whose imaginary parts are all equal and the path of integration a line parallel to the real axis.

The series (3) has this advantage over (1) that it is usually convergent, the conditions of convergence being similar to those of a Fourier series. It is however in general discontinuous; for, when the lower limits are all made equal to zero, then if

$$\begin{aligned}
x &= 0, & f(0) &= -\tfrac{1}{2}g(0),\\
0 \leqq x &\leqq 1, & f(x) &= 0,\\
x &= 1, & f(1) &= \tfrac{1}{2}g(0),\\
1 \leqq x &\leqq 2, & f(x) &= g(x-1),\\
&\cdots\cdots\cdots & &\\
n \leqq x &\leqq n+1, & f(x) &= g(x-1) + g(x-2) + \cdots g(x-n).
\end{aligned}$$

As the derivation of the series (3) is purely formal, it is necessary to verify directly that it does satisfy the equation $\Delta f(x) = g(x)$. Forming

$\Delta f(x)$ we have

$$\Delta f(x) = -\tfrac{1}{2}[g(x+1) - g(x)] + \int_x^{x+1} g(t)dt + 2\sum_1^\infty \int_x^{x+1} \cos 2\pi n(x-t)g(t)dt$$

$$= -\tfrac{1}{2}[g(x+1) - g(x)] + \tfrac{1}{2}[g(x+1) + g(x)] = g(x),$$

since the series

$$\int_x^{x+1} g(t)dt + 2\sum_1^\infty \int_x^{x+1} \cos 2\pi n(x-t)g(t)dt$$

is the value of the Fourier series for $g(x)$ at the end of the range of integration. Hence (3) gives a value of $\Delta^{-1}g(x)$ when $g(x)$ is a continuous function capable of being represented by a Fourier series.

This solution is of course not essentially different from the sum formula of Poisson: it is also closely related to the Guichard integral from which it may be derived.

2. As an illustration of the use of our form of the solution we shall now derive from it the well known asymptotic series for $\log \Gamma(x)$ which is a solution of the equation

(4) $$\Delta f(x) = \log x.$$

Forming the series for $f(x)$, we obtain as a solution of (4)

$$f_1(x) = -\tfrac{1}{2}\log x + \int_1^x \log t dt + 2\sum_1^\infty \int_1^x \cos 2\pi n(x-t)\log t dt$$

$$= (x - \tfrac{1}{2})\log x - x + 1 + 2\sum_1^\infty - \left[\frac{\sin 2\pi n(x-t)}{2\pi n}\log t\right]_1^x$$

$$+ 2\sum_1^\infty \int_1^x \frac{\sin 2\pi n(x-t)}{2\pi nt}$$

$$= (x - \tfrac{1}{2})\log x - x + 1 + 2\sum_1^\infty \int_1^x \frac{\sin 2\pi n(x-t)}{2\pi nt} dt.$$

If we now change the lower limit of integration from 1 to ∞, the value of the series is only changed by a periodic function of period unity, so that

$$f_2(x) = (x - \tfrac{1}{2})\log x - x + k - 2\sum_1^\infty \int_x^\infty \frac{\sin 2\pi n(x-t)}{2\pi nt} dt$$

$$= (x - \tfrac{1}{2})\log x - x + k + 2\sum_1^\infty \frac{1}{(2\pi n)^2} x^{-1} + 2\sum_1^\infty \int_x^\infty \frac{\cos 2\pi n(x-t)}{(2\pi nt)^2 t^2} dt,$$

where k is a constant, is a solution provided the series used are convergent.

When x is real and positive,

$$|S| \equiv \left| \sum_1^\infty \int_x^\infty \frac{\cos 2\pi n(x-t)}{(2\pi n)^2 t^2}\, dt \right| \leqq \sum_1^\infty \int_x^\infty \frac{dt}{(2\pi n)^2 t^2} = \frac{1}{x} \sum_1^\infty \frac{1}{(2\pi n)^2},$$

while, if $x = \xi + \eta i$, $t = \alpha + \eta i$, $(\eta \neq 0)$,

$$|S| \leqq \sum_1^\infty \int_\xi^\infty \frac{d\alpha}{(2\pi n)^2(\alpha^2 + \eta^2)} \leqq \frac{\pi}{|\eta|} \sum_1^\infty \frac{1}{(2\pi n)^2}.$$

The series therefore converges and

$$f_2(x) \sim k + (x - \tfrac{1}{2}) \log x - x + \frac{2}{x} \sum_1^\infty \frac{1}{(2\pi n)^2}$$

so long as x does not lie on the negative half of the real axis. It is easily shown by the usual methods that k must equal $\frac{1}{2} \log 2\pi$ in order that $f_2(x)$ may coincide with $\log \Gamma(x)$.

3. Another solution of the simple difference equation may be obtained by means of the integral equation

$$\int^x g(t)dt = \int_x^{x+1} f(t)dt, \tag{7}$$

which on differentiation leads to

$$g(x) = f(x+1) - f(x).$$

As is suggested by the solution of the Fredholm equation, we consider first the homogeneous equation

$$u(x) = \lambda \int_x^{x+1} u(t)dt = \lambda \int_a^\infty K(x, t)u(t)dt, \tag{8}$$

where the kernel $K(x, t)$ is defined by

$$\begin{aligned} K(x, t) &= 0, \quad \alpha \leqq t < x \\ &= 1, \quad x \leqq t \leqq x+1 \\ &= 0, \quad x+1 < t, \end{aligned}$$

α being a fixed constant. This kernel has a continuous spectrum: for if in (8) we substitute

$$u(x) = e^{kx}, \qquad (k \neq 2n\pi i) \tag{9}$$

we get, after a simple reduction,

$$\lambda = \frac{k}{e^k - 1}, \tag{10}$$

so that λ may have any finite value.

The theory of integral equations therefore suggests the expansion of $\int^x g(t)dt$ in a series of the form

$$\int^x g(t)dt = \Sigma a_i e^{k_i x}. \tag{11}$$

If an expansion of this form exists, a solution of (7) is

$$f(x) = \Sigma a_i \frac{k_i}{e^{k_i} - 1} e^{k_i x}, \tag{12}$$

provided this series is convergent.

If $g(x)$ can be expanded in a uniformly convergent series

$$g(x) = \Sigma b_i e^{k_i x}, \tag{13}$$

then $b_i = a_i k_i$, and we may set for $f(x)$

$$f(x) = \Sigma \frac{b_i}{e^{k_i} - 1} e^{k_i x} + \omega(x). \tag{14}$$

When $g(x)$ is a real function which satisfies Dirichlet's conditions, an expansion of the form (13) always exists. In particular we have*

$$g(x) = \Sigma \frac{e^{a\sigma i}(\sigma i + h)}{\sin 2a\sigma - 2a\sigma} \sin a\sigma \cdot e^{\sigma i x} \int_{-a}^{a} e^{-\sigma i t} g(t)dt, \tag{15}$$

where the summation extends over all the roots of

$$\sigma + h \tan a\sigma = 0, \tag{16}$$

and $-a < x < a$.

None of the roots of (16) make $e^{\sigma i} - 1 = 0$, and as σ approaches infinity, σ approaches the value $(2n + 1)\pi/2a$, so that if a is an integer, $e^{\sigma i} - 1$ remains greater than some finite quantity. Hence the form of $f(x)$ given by (14) is convergent and a solution of (7) is therefore given by

$$\begin{aligned} f(x) &= \frac{hx}{2(1+ah)} \int_{-a}^{a} g(t)dt + \Sigma \frac{e^{a\sigma i}(\sigma i + h) \sin a\sigma}{(\sin 2a\sigma - 2a\sigma)(e^{\sigma i} - 1)} \int_{-a}^{a} e^{\sigma i(x-t)} g(t)dt + \omega(x) \\ &= \frac{hx}{2(1+ah)} \int_{-a}^{a} g(t)dt + \Sigma \frac{\sigma}{2a\sigma - \sin 2a\sigma} \int_{-a}^{a} g(t) \frac{\sin \sigma(x - t - \frac{1}{2})}{\sin \frac{1}{2}\sigma} dt + \omega(x), \end{aligned}$$

where the second summation extends only over positive values of σ.

* See for instance Picard, Traité d'Analyse, 2e ed., t. 2, p. 192.

NOTE ON THE RANK OF A SYMMETRICAL MATRIX. II.

By J. H. M. Wedderburn.

In an earlier paper with the same title, I gave a proof of the theorem,* that in a symmetrical matrix of rank r there is a principal minor of order r which does not vanish, by first showing that the sum of the principal minors of order r is not zero. This is however not necessarily true when the coefficients of the matrix are complex as is seen from the example†

$$\begin{Vmatrix} 1 & i \\ i & -1 \end{Vmatrix}.$$

The origin of this exceptional case lies in the fact that it is not possible to find an orthogonal matrix which will transform the point, or vector, $(x_{11}, x_{12}, \cdots x_{1n})$ into $(1, 0, \cdots 0)$ when

$$\sum_{s=1}^{n} x_{1s}^2 = 0. \tag{1}$$

It is however easily shown by induction that the required orthogonal matrix does exist when (1) is not satisfied. In extending the theorem to complex coefficients we have therefore only to consider the effect of null vectors of the given matrix whose tensors are zero.

If the tensor of‡

$$x + iy = (\xi_1 + i\eta_1, \xi_2 + i\eta_2, \cdots \xi_n + i\eta_n)$$

is zero, we have

$$\sum_{s=1}^{s=n} \xi_s^2 = \sum_{s=1}^{s=n} \eta_s^2, \qquad \sum_{s=1}^{s=n} \xi_s \eta_s = 0.$$

The two vectors, $x = (\xi_1, \xi_2, \cdots \xi_n)$ and $y = (\eta_1, \eta_2, \cdots \eta_n)$, are therefore orthogonal and, seeing that their tensors are equal and not zero, there exists an orthogonal matrix which transforms them into e_1 and e_2 respectively, e_s denoting the vector all of whose coefficients are zero except the sth which is 1.

Suppose now that $x + iy$ is a null vector of a matrix, A, whose rank is r. If A is transformed by the orthogonal matrix determined above,

* Frobenius, "Über das Pfaffsche Problem," Crelle, vol. 82, 1877, p. 242.

† This example is due to Frobenius.

‡ By the sum, $x + y$, of two vectors is meant the vector obtained by adding corresponding coefficients of x and y.

the transformed matrix has $e_1 + ie_2$ as a null vector and therefore has the form

$$A_1 \equiv \begin{Vmatrix} \alpha & i\alpha & \beta_3 & \beta_4 & \cdots & \beta_n \\ i\alpha & -\alpha & i\beta_3 & i\beta_4 & \cdots & i\beta_n \\ \beta_3 & i\beta_3 & a_{33} & a_{34} & \cdots & a_{3n} \\ \beta_4 & i\beta_4 & a_{34} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_n & i\beta_n & a_{3n} & \cdot & \cdot & \cdot \end{Vmatrix}.$$

A_1 has the same rank as the matrix

$$A_2 \equiv \begin{Vmatrix} \alpha & \beta_3 & \beta_4 & \cdots & \beta_n \\ \beta_3 & a_{33} & a_{34} & \cdots & a_{3n} \\ \beta_4 & a_{34} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_n & a_{3n} & \cdot & \cdot & \cdot \end{Vmatrix},$$

which is symmetric and of order $n - 1$: for, any minor of A_1 which does not contain any element of the second row or column is also a minor of A_2; any minor which contains elements of both the first and the second row (column) is zero; and one which contains the elements of the second but not of the first row (column), and does not come under the case just mentioned, is i times the minor obtained by replacing the second row (column) by the first.

Since principal minors of A_2 are also principal minors of A_1 and the order of A_2 is one less than the order of A_1 while the ranks are the same, the fact that one principal minor of order r does not vanish follows readily by induction.

The proof may also be made by transforming A_1 by the non-singular matrix

$$C \equiv \begin{Vmatrix} 1 & 0 & & & & \\ i & 1 & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{Vmatrix},$$

which gives

$$C'A_1C = \begin{Vmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & -\alpha & i\beta_1 & \cdots & i\beta_n \\ 0 & i\beta_1 & a_{33} & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & i\beta_n & a_{3n} & \cdots & a_{nn} \end{Vmatrix}.$$

In conclusion we may notice that the characteristic equation of A_1 is the same as that of the matrix obtained by replacing the first and second rows of A_1 by zeros.

ERRATA

Volume 15.

On the Projective Differential Geometry of Plane Anharmonic Curves. By Samuel W. Reaves. Professor A. M. Harding has called my attention to the fact that equations (33) and (34) should read

$$(33)\quad 2e^{2t}x^2 + 8e^t xy - y^2 + (10 - 4t)e^{2t}x - (2 + t)8e^t y + (2t^2 - 10t + 17)e^{2t} = 0,$$

$$(34)\qquad y = -2e^{x-3/2}.$$

S. W. R.

NOTE ON NORMAL SECTIONS OF A SURFACE IN A SPACE OF N DIMENSIONS.

By C. L. E. Moore.

1. In his dissertation Karl Kommerell* discussed surfaces (two-dimensional configurations of points) in a space of four dimensions and generalized many of the well-known properties of surfaces in ordinary space. The method used was that of ordinary theory of surfaces. In this note some of these properties are generalized for surfaces in a space of n dimensions. For the properties considered the generalization is complete when $n = 5$. The method here used is that of vector analysis and the notation used is that of Grassmann's Ausdehnungslehre.†

2. If we represent a surface vectorially by the equation

$$x = x(u, v)$$

then the unit tangent vector and curvature vector of a curve traced on the surface are dx/ds and d^2x/ds^2, where s denotes the length of arc of the curve. The differentials dx and d^2x will denote vectors directed respectively along the tangent and the principal normal. The osculating plane of a curve on the surface is defined by

$$[dxd^2x], \tag{1}$$

which on expanding becomes

$$\begin{aligned}[(x_1du + x_2dv)(x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2 + x_1d^2u + x_2d^2v] \\ = [(x_1du + x_2dv)X] + [x_1x_2](dud^2v - dvd^2u),\end{aligned} \tag{2}$$

where

$$x_1 = \frac{\partial x}{\partial u}, \qquad x_2 = \frac{\partial x}{\partial v}, \qquad x_{11} = \frac{\partial^2 x}{\partial u^2}, \qquad \text{etc.},$$

and

$$X = x_{11}du^2 + 2x_{12}dudv + x_{22}dv^2.$$

* "Die Krümmung der zwei dimensionalen Gebilde im ebenen Raum von vier Dimensionen," Tübingen, 1897. A synopsis of the results of this paper is found in an article by the same author, "Riemannsche Flächen im ebenen Raum von vier Dimensionen." Mathematische Annalen, vol. 60, 1905.

† Gesammelte Werke, vol. I. Good expositions of this analysis can also be found in the following places: Encyclopédie, édition Français, Vol. I.; Nombres Complexes, Study-Carton; Universal Algebra, Whitehead; Application de la methode vectorielle, Fehr; Louis Ingold, Transactions of the American Mathematical Society, vol. 11 (1910).

If du/dv is held fixed and d^2u/d^2v is allowed to vary, since the parameter $(dud^2v - dvd^2u)$ enters linearly, we see that the plane (2) generates a space of three dimensions. This S_3 is given by

$$[x_1x_2X]. \tag{3}$$

For values of d^2u/d^2v can be found so that the product of x_1, x_2 or X with the plane (2) will vanish and therefore the S_3 must be that determined by these three vectors. The tangent plane to the surface is $[x_1x_2]$ and is therefore contained in the S_3 above. This is called *the osculating 3-space* and is said to osculate the surface along the direction du/dv. If du/dv also varies, the osculating S_3's will generate a quadric cone of four dimensions, V_4^2, since X is quadratic in du/dv. From equations (2) and (3) we see that this cone lies in the space of five dimensions

$$[x_1x_2x_{11}x_{12}x_{22}], \tag{4}$$

and has the tangent plane for vertex.*

3. The normals to a surface at a point P, that is lines perpendicular to the tangent plane, will generate a linear space of $n - 2$ dimensions, which we will denote by N_{n-2}. The normal space is the complement of the tangent plane $[x_1x_2]$ and according to Grassmann will be denoted by $|[x_1x_2]$. The osculating S_3 along a given direction du/dv will cut N_{n-2} in a line normal to the surface. Hence, *of all the curves traced on a surface which are tangent to a fixed direction at a fixed point there is just one whose osculating plane contains a normal to the surface.*

Let us confine our attention, for the moment, to a given osculating S_3. Call the *unit* normal lying in it n and the unit principal normal to the curve ν. The angle between these two directions is given by the formula

$$[n|\nu] = \cos\theta.$$

From the Frenet formulæ we have

$$\frac{d^2x}{ds^2} = \frac{\nu}{\rho},$$

where ρ denotes the radius of curvature of the curve, and consequently

$$\left[\frac{d^2x}{ds^2}\middle| n\right] = \frac{1}{\rho}[\nu|n] = \frac{\cos\theta}{\rho}.$$

* These results are due to Del Pezzo, "Sugli spazi tangenti ad una superficie o ad una varietá immersa in uno spazio di piu dimensione." Rendiconti Accademia della Scienze di Napoli, 1886. The treatment is entirely synthetic. Segre obtained the same results analytically in a paper, "Su una classa di superficie degl' iperspazi, ecc.," Atti di Torino, 1907.

Expanding the right side of this equation, we have

$$[(x_{11}u'^2 + 2x_{12}u'v' + x_{22}v'^2 + x_1u'' + x_2v'')|n] =$$

$$[(x_{11}u'^2 + 2x_{12}u'v' + x_{22}v'^2)|n] = \frac{\cos\theta}{\rho}.$$

(The accents denote derivatives with respect to the arc length.) The right side of this equality does not depend on the second order derivatives of u and v. Therefore if du/dv is held fixed, that is, if we consider a fixed osculating S_3, we have

$$\frac{\cos\theta}{\rho} = \text{const.} = \frac{1}{R},$$

or

$$\rho = R\cos\theta. \tag{5}$$

Here R denotes the radius of curvature of the curve whose osculating plane contains the normal n of the surface, that is the radius of curvature of the normal section of the surface corresponding to the direction du/dv. Meusnier's theorem for surfaces in space of n dimensions is then: *On a surface in S_n the osculating circles to curves passing through a fixed point and tangent to a fixed direction generate a sphere tangent to the surface.*

These results are readily generalized for any variety V_m in S_n. The osculating planes of curves tangent to a fixed direction generate an S_{m+1} which contains the tangent S_m. There is just one of these curves whose principal normal is normal to the variety. In this S_{m+1} Meusnier's theorem is the same as for a V_m in S_{m+1}.

4. Corresponding to each direction on the surface there is one curve whose principal normal is normal to the surface. This one parameter family of normals, corresponding to the different directions traced on the surface, will generate a quadric cone. For the osculating S_3's generate a cone V_4^2. This cone is cut by the normal space N_{n-2} in a quadric cone lying in the S_3 formed by the intersection of N_{n-2} and the S_5, $[x_1x_2x_{11}x_{12}x_{22}]$. Hence, *the curvature vectors of curves traced on a surface through a fixed point and whose osculating planes are normal to the surface, generate a quadric cone lying in an S_3.* We will call these curves normal sections of the surface.

From this it is seen that so far as curvature is concerned we have the complete generalization when we consider surfaces in S_5. This would follow from the fact also that the points of a surface near a fixed point to infinitesimals of second order all lie in a space of five dimensions.

If the cone V_4^2 degenerates, that is if

$$[x_1x_2x_{11}x_{12}x_{22}] = 0, \tag{6}$$

the osculating S_3's generate an S_4, and the curvature vectors of the normal sections will generate a plane pencil lying in the plane of intersection of this S_4 with N_{n-2}.

Segre* has developed the projective theory of the surfaces whose coördinates satisfy relation (6) at each point. So far as curvature is concerned it is evident that these surfaces have a very close resemblance to surfaces in a space of four dimensions. Kommerell showed that in S_4 an osculating S_3 (or in this case any hyperplane passed through the tangent plane) osculates the surface in two different directions. We will now show that this is also true for surfaces in S_n which satisfy relation (6). The osculating S_3 is

$$[x_1x_2(x_{11}\tau^2 + 2x_{12}\tau + x_{22})] \qquad \tau = du/dv,$$

and (6) is equivalent to

$$(C) \qquad Ax_{11} + 2Bx_{12} + Cx_{22} + Dx_1 + Ex_2 = 0.$$

Eliminating x_{22}, for example, we have

$$(3') \qquad [x_1x_2x_{11}]\left(\tau^2 - \frac{A}{c}\right) + [x_1x_2x_{12}]\left(\tau - \frac{B}{c}\right) \qquad \text{(for } \tau \text{ fixed).}$$

Any osculating S_3 can be put into the form

$$[x_1x_2x_{11}]\left(t^2 - \frac{A}{c}\right) + [x_1x_2x_{12}]\left(t - \frac{B}{c}\right) \quad \text{(for } t \text{ variable).}$$

The second will coincide with the first, if

$$(7) \qquad \frac{Ct^2 - A}{C\tau^2 - A} = \frac{Ct - B}{C\tau - B}.$$

Since this relation is quadratic in t there are two directions along which it will osculate the surface. It is easily verified that these two directions will coincide if

$$(8) \qquad C\tau^2 - 2B\tau + A = 0.$$

This is the equation which defines the characteristics of the equation (C). Equation $(3')$ shows that any S_3 in the S_4 defined by $[x_1x_2x_{11}x_{12}x_{22}] = 0$ and containing the tangent plane is an osculating S_3. If then we write any osculating S_3 in the form

$$[x_1x_2x_{11}]M + [x_1x_2x_{12}]N,$$

the directions along which it osculates the surface are given by

$$\frac{Ct^2 - A}{M} = \frac{Ct - B}{N}.$$

* Loc. cit.

It is easily seen that these two directions are harmonic with respect to the two directions defined by (8). Two directions along which the same S_3 osculates are called conjugate directions. *The directions of the characteristics are self-conjugate and are separated harmonically by each pair of conjugate directions.*

If not only (6) but also equations

$$[x_1x_2x_{11}x_{12}] = 0, \qquad [x_1x_2x_{11}x_{22}] = 0 \tag{9}$$

are satisfied, the same S_3 will osculate in all directions. For in this case the five vectors x_1, x_2, x_{11}, x_{12}, x_{22} lie in an S_3. The normal is the intersection of this S_3 and N_{n-2}. All the normal sections of the surface determine the same normal to the surface. Segre* has shown that if two such equations are satisfied, and n is greater than three the surface must be a developable, that is formed by the tangent lines to a twisted curve. These surfaces then, so far as curvature is concerned, behave like surfaces in ordinary space.

5. If we make u and v functions of a single variable t, the magnitude of the curvature of the curve thus obtained is

$$c = \frac{\sqrt{[x''x'][x''x']}}{[x'|x']^{\frac{3}{2}}}, \tag{11}$$

accents denoting differentiation with respect to t. For let τ denote the unit tangent, then

$$\frac{dx}{ds} = \tau = \frac{dx}{dt}\frac{dt}{ds} = \frac{x'}{s'}, \qquad s' = \frac{ds}{dt},$$

$$x' = s'\tau, \qquad x'' = s''\tau + s'\tau'.$$

Then

$$[x'x''] = [s'\tau(s''\tau + s'\tau)] = s'^2[\tau\tau'].$$

But

$$c = \frac{d\tau}{ds} = \frac{\tau'}{s'},$$

or

$$\tau' = s'\mathbf{c}.$$

$\mathbf{c}$ denotes the curvature vector. Substituting this value, we have

$$[x'x''] = s'^2[x'\mathbf{c}].$$

Taking the inner product of this with itself, we have

$$[x'x''][x'x''] = s'^4[x'\mathbf{c}][x'\mathbf{c}] = s'^6\mathbf{c}^2,$$

* Loc. cit.

or

$$c = \frac{\sqrt{[x_1 x'']|[x' x'']}}{s'^3}.$$

But $s'^2 = [x'|x']$, which substituted gives formula (11). If we take $z_1, z_2, \cdots z_n$ as the components of the vector x we have

$$c = \frac{1}{\rho} = \frac{\sqrt{\Sigma(z_i' z_j'' - z_j' z_i'')^2}}{\{\Sigma z_i'^2\}^{\frac{3}{2}}}.$$

If we take the origin on the surface and take the z_1, z_2 axes tangent to the parameter curves, then the components of the vector x can be written in the form

$$\begin{aligned} z_1 &= u, \quad z_2 = v, \\ z_3 &= a_3 u^2 + 2b_3 uv + c_3 v^2 + \cdots, \\ &\cdots\cdots\cdots\cdots \\ z_n &= a_n u^2 + 2b_n uv + c_n v^2 + \cdots. \end{aligned} \tag{12}$$

A curve on the surface can now be defined by

$$v = \lambda u. \tag{13}$$

This is, in fact, a curve cut from the surface by a hyperplane containing the normal N_{n-2}. For such a curve we then have at the origin

$$z_1 = z_2 = z_3 = \cdots z_n = 0. \qquad \frac{dv}{du} = \lambda, \qquad \frac{d^2 v}{du^2} = 0.$$

Using these values the formula for the curvature becomes

$$c^2 = \Sigma\left(\frac{a_i + 2b_i\lambda + c_i\lambda^2}{1 + \lambda^2}\right)^2, \qquad (i = 3, 4, \cdots n). \tag{14}$$

Putting in the condition for a maximum or minimum, namely

$$\frac{dc^2}{d\lambda} = 0,$$

we have

$$\Sigma(a_i + 2b_i\lambda + c_i\lambda^2)(a_i\lambda^2 + (a_i - c_i)\lambda - b_i) = 0.$$

Since this equation is of the fourth degree in λ there are four directions of maximum or minimum curvature.

Kommerell* pointed out that the asymptotic lines on an ordinary surface in 3-space satisfy this condition. The theorem that through each point on a surface in any number of dimensions pass four lines of

* Loc. cit.

curvature, is general if we admit the asymptotic lines in ordinary space. The right side of (14) in the case of ordinary space has a single term

$$c = \frac{a_3 + 2b_3\lambda + c_3\lambda^2}{1 + \lambda^2},$$

and in this case there are real values of λ which will make c zero if $b_3^2 - a_3c_3 > 0$. But for $n > 3$ we see that there are no real values of λ which will make the curvature zero on a real surface.

6. To every direction on the surface corresponds a direction in N_{n-2}. This is the intersection of the osculating S_3 with N_{n-2}. Expressed as a product this is

$$[x_1x_2X][x_1x_2] = [x_1x_2X][z_3z_4 \cdots z_n]. \tag{15}$$

The complement of the tangent plane $[x_1x_2]$ is $[z_3z_4 \cdots z_n]$, since x_1, x_2 coincide with z_1, z_2. Developing this product according to elements of the last factor, we have

$$[x_1x_2Xz_4 \cdots z_5]z_3 - [x_1x_2Xz_3z_5 \cdots z_n]z_4 + [x_1x_2Xz_3z_4z_6 \cdots z_n]z_5 + \cdots.$$

The directions of the normal corresponding to the direction du/dv are then proportional to the coefficients of the z's in this vector. From equations (12) and (13) we have

$$[x_1x_2Xz_4z_5 \cdots z_n] = \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots & A \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & A_4 & \cdots & A_n \\ 0 & 0 & 0 & 1 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = A_3,$$

where $A_i = a_i + 2b_i\lambda + c_i\lambda^2$. Making use of (14), we can put the normal curvature vector in the form

$$c = \frac{1}{1 + \lambda^2}(A_3z_3 + A_4z_4 + \cdots + A_nz_n].$$

The locus of the end of the vector is then,

$$z_i = c \cos \alpha_i,$$

where c is the length of the curvature vector and α_i the angle which it makes with the z_i-axis. The parametric equations of the locus are then,

$$z_i = \frac{\sqrt{\Sigma A_i^2}}{1 + \lambda^2} \cdot \frac{A_i}{\sqrt{\Sigma A_i^2}} = \frac{A_i}{1 + \lambda^2} = \frac{a_i + 2b_i\lambda + c_i\lambda^2}{1 + \lambda^2}. \tag{16}$$

Therefore, *the locus of the end of the normal curvature vector is a conic.*

The radius of curvature is $1/c$. Hence *the locus of the centers of normal curvature is the inverse of the above conic with respect to a unit circle and is therefore a twisted quartic lying on the quadric cone generated by the normal curvature vectors.*

From equation (14) we saw that c could neither be zero nor infinite for a real direction on the surface. The conic (16) must then be an ellipse and cannot pass through the point on the surface. For a real point on the surface there are two imaginary directions for which the curvature is infinite or the radius of curvature zero. But for an ordinary real point there is no direction for which the curvature is zero. Equation (14) however shows that there are directions such that the magnitude of the curvature is zero. This would be the case if

$$\Sigma A_i^2 = 0.$$

That is if the magnitude of the curvature is zero, the curve has the direction of a minimum line. Equation (14) however shows that the corresponding point of the locus (16) will not be at the origin unless $A_i = 0$, which would not be the case for a generic point of a general surface. If this last relation is satisfied however, there is a direction such that the curvature of a normal section is zero which would mean that this direction has three point contact with the surface.

If the coördinates of the surface satisfy equation (6) the cone of normal radii of curvature will degenerate into a plane but the conic (16) will not degenerate. These surfaces then behave like a surface in S_4.

From the fact that the end of the curvature vector traces a conic can be seen at once that there are four maxima and minima for **c** and hence passing through each point of the surface are four lines of curvature.

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AN ALGEBRAIC TREATMENT OF THE THEOREM OF CLOSURE.

By Albert A. Bennett.

Introduction.

1. Among the classic theorems concerning the projective properties of a pair of conics, perhaps the most interesting is one due to Poncelet, viz. the theorem that if a polygon of n sides can be circumscribed about one conic and at the same time inscribed in a second conic, it is possible to construct an infinite number of such polygons for the given pair of conics. A very elegant demonstration of this theorem may be made by the use of elliptic functions, but a parallel algebraic treatment is also possible. From an algebraic point of view, we have here but one example of a certain interesting class of problems in elimination. We shall mention the general algebraic problem, but shall carry through the details only in the hyperelliptic case. Except in so far as is necessary to make the algebraic steps clear no discussion will be made of the numerous geometric corollaries that suggest themselves.

The present treatment is an attempt to reduce the problem to its simplest form and to prove the theorems needed with a minimum of algebraic machinery. Little emphasis is placed upon the numerous features which serve to individualize the elliptic within the general hyperelliptic problem.

The functions considered are those well-known in the transcendental theory, although the methods of proof are of necessity largely new. Constant use has been made of the remarkably clearly written Traité des Fonctions Elliptiques by Halphen.

It should be noted that not only are the operations used in this paper algebraic, but that except for a single irrationality, $\sqrt{\Delta}$, every step is essentially rational. Neither the notions of geometric continuity nor of convergence of series are required at any stage. Thus the present discussion is applicable in its entirety to finite fields, a statement which does not hold true of the algebraic treatments already published.

Extensive references to the literature on the problem of closure in the elliptic case may be found in the Encyklopädie der Math. Wiss., III, C 1, p. 45 ff., the Encyk. der Geometrie (Simon), p. 105 ff. and in Pascal's Repertorium, II^{1}, p. 238 ff.

Modern algebraic treatments of the Poncelet Polygons are given by

K. Petr, "Ponceletsche Polygone," Monatsh. f. Math. u. Phys. 18, (1907), pp. 122–137, and Karl Rohn, "Das Schliessungsproblem von Poncelet . . . ," Berichte d. Ges. der Wiss. zu Leipzig, 60 (1908), p. 94. The present treatment differs in aim from these and has little in common with them.

A very admirable summary of the principal theorems upon groups of points on an algebraic curve is found in Pascal's Repertorium, II[1], pp. 306–355. A good treatise of an elementary sort is Severi's Lezioni di Geometria Algebrica.

An Arithmetic of Points on a Cubic Curve.

2. Any cubic curve, with or without a double point, but not resolvable into curves of lower order, can be reduced by the use of projective transformations alone to the form

$$y^2 = x^3 + \delta_1 x + \delta_2$$

where δ_1 and δ_2 are constants and δ_1^3/δ_2^2 is an absolute invariant.

Any polynomial of the form

$$x^3 + \delta_1 x + \delta_2,$$

we shall call $\Delta(x)$, and the cubic curve defined by $y^2 = \Delta(x)$, we shall call a curve K. Any point on a curve K we shall call a point P. Any linear equation in x and y we shall call an equation L, and the line defined by it we shall call a line L. This notation is adopted in order to facilitate a generalization from the case of the cubic curve to that of the hyperelliptic curve, and the same notation will be used in both cases.

3. If a curve K be written in the form $y^2 = \Delta(x)$, then with respect to this curve and this method of representation, a fixed set of finite points on the curve will be said to be in " general position," if no two of the points have the same x-coordinates, i. e., if no two of the points are either coincident or are obtainable one from the other by a mere reflexion in the x-axis. A variable set of points is said to be in " general position," if, except for a zero dimensional set of cases, every particular position of the variable set is in general position according to the previous definition.

We shall have occasion to use the following theorems:

(*a*) Any two finite P's in general position on K lie on one and only one L. This L intersects K again in a uniquely determined P.

(*b*) There is at least one P on K, such that an L tangent to K at one of these P's, has there contact of the second order, in other words, does not meet K at any other distinct point. Such P's are called " Inflexional."

(*c*) If K be non-degenerate and fixed, and F' and F'' be any two curves, degenerate or not, fixed or variable, but of the same degree as K and such

that the finite points of intersection of F' with K consist of a set S together with a single P, e. g., P_1, and the finite points of intersection of F'' with K consist of the same set S together with a single P, e. g., P_2, while S is in general position on K; then P_1 coincides with P_2. We here suppose that the common points of intersection of any two of the curves K, F', and F'' shall be at most a zero-dimensional set of points.

(*d*) The P which lies at infinity we shall call P_0. If an L passes through P_0 and through a finite point on K, the equation of L is satisfied by the line which is drawn through the finite point and parallel to the y-axis. So that any L through P_0 meeting K in a finite P, meets it also in a second P obtainable from the first by a reflexion in the x-axis. Such a reflexion carries an inflexional P into an inflexional P.

Property (*c*) states what is often put in the form, "The K's through eight P's pass also through a ninth," and (*d*) includes a special case of the theorem that an L through two inflexional P's passes through a third.

We shall later prove each of these properties (*a*), (*b*), (*c*), (*d*).

4. If three P's, e. g., P_α, P_β, P_γ, of K are on a single L, we shall indicate this relation by the symbolic equation

$$P_\alpha + P_\beta + P_\gamma = 0.$$

The P obtained from P_α by reflexion in the x-axis, we shall indicate by $-P_\alpha$ or $P_{-\alpha}$, so that we may write

$$P_\alpha + (-P_\alpha) + P_0 = 0.$$

If a P_γ and a $P_{\gamma'}$ are so related that one is the negative of the other, and if furthermore,

$$P_\alpha + P_\beta + P_{\gamma'} = 0,$$

then we shall indicate the relation of P_γ to P_α and P_β by the symbolic equation

$$P_\gamma = P_\alpha + P_\beta,$$

which yields upon reflexion in the x-axis,

$$-P_\gamma = (-P_\alpha) + (-P_\beta).$$

Furthermore,

$$P_\alpha = P_\alpha + P_0,$$

and in particular

$$P_0 = P_0 + P_0.$$

We have thus defined a sort of addition which is obviously commutative, we shall now show that it is also associative.

5. Let us take a particular nondegenerate curve K. We will take on K, three arbitrary P's, e. g., P_α, P_β, P_γ. Through P_α and P_β, there passes

an L, which we shall call L_1', and through P_β and P_γ, another L, viz., L_1''. Let the third P, in which L_1' meets K, be designated by $-P_\delta$, and the third P in which L_1'' meets K, by $-P_\epsilon$. We shall join $-P_\delta$, and $-P_\epsilon$ respectively with P_0, by L_3'' and L_3' respectively. These will determine P_δ and P_ϵ, respectively, as the third P of intersection. Let L_2'' join P_α and P_ϵ, and L_2' join P_γ with P_δ. We shall prove that the third P on L_2'' coincides with the third P on L_2', and this we shall represent by $-P_\zeta$. The incidence relations may be summarized by the following table:

	L_1'	L_2'	L_3'
L_1''	P_β	P_γ	$-P_\epsilon$
L_2''	P_α	$-P_\zeta$	P_ϵ
L_3''	$-P_\delta$	P_δ	P_0

We may suppose P_α, P_β, P_γ, P_δ, P_ϵ, $-P_\delta$, $-P_\epsilon$ to be finite and in general position on K. Now L_1', L_2', L_3' taken together constitute a degenerate curve F', and L_1'', L_2'', L_3'' constitute another degenerate curve F''. Hence if we define as $-P_\zeta$, the P which is the remaining intersection of L_2' with K, then by property (c) mentioned above, L_2'' also must intersect K in this same $-P_\zeta$. Thus the above incidence table is justified. It holds also for special positions, if the L's exist.

In terms of the symbolic equations, we shall therefore have

$$\begin{aligned} P_\delta &= P_\alpha + P_\beta, \\ P_\epsilon &= P_\beta + P_\gamma, \\ P_\zeta &= P_\alpha + P_\epsilon, \\ P_\zeta &= P_\delta + P_\gamma. \end{aligned}$$

$$\therefore \quad P_\alpha + (P_\beta + P_\gamma) = (P_\alpha + P_\beta) + P_\gamma.$$

Hence this addition is associative.

6. Let us now take an arbitrary fixed P_α and any P_ξ. Whenever $P_\xi + P_\alpha$ exists, we may define $P_{\xi+\alpha}$ by the equation

$$P_{\xi+\alpha} = P_\xi + P_\alpha.$$

Similarly $P_{\xi+2\alpha} = P_{\xi+\alpha} + P_\alpha$, and in general $P_{\xi+n\alpha} = P_{\xi+(n-1)\alpha} + P_\alpha$, where n is any positive integer. If now for a particular ξ and n, we have

$$P_{\xi+n\alpha} = P_\xi,$$

then by adding $-P_\xi$ to each member we obtain

$$P_{n\alpha} = P_0,$$

which is independent of the P_ξ chosen originally. Hence, we have the so-called

Theorem of Closure. *If for a given fixed* P_a *and a given constant integer* n, *and for a single choice of* P_ξ, *we should obtain* $P_{\xi+n\alpha} = P_\xi$, *then the same is true for every choice of* P_ξ.

Now $P_{n\alpha}$ may be determined algebraically from P_α, and since the determination is unique, $P_{n\alpha}$ is obtainable rationally from P_α. The condition imposed upon P_α in order that $P_{n\alpha} = P_0$ is algebraic, and is known as the "Condition of Closure of the nth Order."

Two-Two Correspondences.

7. Suppose there be given any two-to-two algebraic correspondence between two one-dimensional projective forms.

For example, consider the correspondence between the points of a nondegenerate point conic, and the lines of a non-degenerate line-conic lying in the same plane, when we define the correspondence as being between incident points and lines. For any point on the first conic, we have two lines through this point and tangent to the second conic, and for any line of the second conic, we have two points on this line, and lying also on the first conic. This is of course the case of the Poncelet Polygons.

Any two-two algebraic correspondence between x and M/N, can be written in the form

$$A(x)M^2 + 2B(x)\,MN + \Gamma(x)\,N^2 = 0,$$

when A, B, and Γ are polynomials in x of the second degree*.

8. Let us now start with a correspondence of the form

$$AM^2 + 2BMN + \Gamma N^2 = 0. \tag{1}$$

Most of the important properties of this correspondence do not depend primarily upon the exact numerical values of the coefficients, but, on the other hand, are unaltered when we replace (1), by a correspondence obtained from it by either or both of the following two transformations:

(a) Replacing M and N by two linearly independent homogeneous linear combinations of them.

(b) Replacing x by a non-singular linear fractional function of x. For the sake of definiteness we may use the freedom at our disposal to reduce (1), to a certain normal form. The discriminant of (1), regarded as a quadratic in M/N, is $B^2 - A\Gamma$, which is invariant under (a). It is a polynomial in x, and by means of (b), we may reduce the leading coefficient to zero, the next to unity and the one following to zero. Thus we obtain an expression of the form which we have already called $\Delta(x)$. We shall

* In the elliptic case M and N are independent of x, but in the hyperelliptic case they are polynomials in x, as explained in § 23.

hereafter write $\Delta \equiv B^2 - A\Gamma$. We still have three parameters at our disposal by using (a), and these we may choose in such a way as to reduce the degree of A by one, and the degree of B by two. We may at the same time divide the whole equation (1) by a constant so as to secure that the now leading coefficient of A is unity, and hence as may be seen from the form of Δ, that the leading coefficient of Γ is minus unity, while the second coefficient of Γ becomes equal to the at present second coefficient of A. There are special cases in which this reduction is not possible in just the form stated, as for instance when Δ or A has a pair of coincident roots. We shall however for the present exclude such cases. All the significant coefficients of A, B, Γ, Δ, when these polynomials are expressed in descending powers of x, may be indicated by the following table:

$$\begin{array}{lllll} A; & 0, & 1, & \alpha_1 & \\ B; & 0, & 0, & \beta_1 & \\ \Gamma; & -1, & \alpha_1, & \gamma_1 & \\ \Delta; & 0, & 1, & 0, & \delta_1. \end{array}$$

Thus we have imposed conditions upon precisely six of the coefficients in the correspondence (1), and have done so in such a manner that $\Delta \equiv B^2 - A\Gamma$ assumes as leading coefficients the numbers indicated in the above scheme.

9. We may write (1) in the irrational form

$$A\frac{M}{N} + B = \pm\sqrt{\Delta}. \tag{2}$$

This suggests that we consider the following two curves or equations,

(a) The fixed K, whose equation is $y^2 = \Delta(x)$,

(b) The variable L, whose equation is $L = 0$, where $L \equiv AM + BN - Ny$, is a rational integral function of x and y.

The variable L meets K in the variable points whose x-coördinates are the roots of (1), considered as an equation in x. But there are also certain fixed points of intersection. For when we eliminate y between the equations for K and for L, every solution of the equation in x so obtained, must be the x-coördinate of a point of intersection of K and L. To eliminate y between the equations for K and L is equivalent to rationalizing (2). When we rationalize (2), we at first obtain, not (1), but

$$A^2M^2 + 2ABMN + B^2N^2 - \Delta N^2 = 0,$$

which differs from (1), by containing throughout the left-hand member, the factor A. When $A = 0$, the equation for the curve K, $y^2 = \Delta$ $(\equiv B^2 - A\Gamma)$ gives $y = +B$ or $-B$, whereas the equation for L gives

$y = \mathrm{B}$. Thus L and K intersect in a fixed P, for which $\mathrm{A} = 0$, $y = \mathrm{B}$, in addition to the variable points of intersection already mentioned.

Not only is this fixed P of intersection determined by A and B, but conversely we may choose a finite P arbitrarily on K and find a unique A and B so as to obtain a correspondence given in the normal form of (1). Indeed from the degree and form of the normalized A and B, respectively, we see that these expressions are uniquely determined by the condition that for a given P, $\mathrm{A} = 0$, $y = \mathrm{B}$. The A and B so determined are such that $\Delta - \mathrm{B}^2$ is divisible by A, for $\Delta - \mathrm{B}^2$ must vanish for $\mathrm{A} = 0$, since this P is also on $y^2 = \Delta$. Hence we may obtain a unique Γ such that $\Delta \equiv \mathrm{B}^2 - \mathrm{A}\Gamma$. With these expressions for A, B, Γ, the correspondence (1) will be in the normal form, and will have the given discriminant Δ.

10. Suppose we be given the curve K and also P_α and P_β selected arbitrarily upon it. Let us consider the rational algebraic process by which $P_{\alpha+\beta}$ may be determined from P_α and P_β.

In place of P_α we may consider the pair of expressions $(\mathrm{A}_\alpha, \mathrm{B}_\alpha)$ such that for P_α, $\mathrm{A}_\alpha = 0$, $y = \mathrm{B}_\alpha$, and similarly we shall replace P_β in our work by $(\mathrm{A}_\beta, \mathrm{B}_\beta)$. The polynomial L in x and y, which vanishes for P_α and for P_β, may be written in either of the following ways:

$$\mathrm{A}_\alpha \mathrm{M} + \mathrm{B}_\alpha \mathrm{N} - \mathrm{N}y, \quad \text{or} \quad \mathrm{A}_\beta \mathrm{M}' + \mathrm{B}_\beta \mathrm{N} - \mathrm{N}y,$$

where N may obviously be taken as the same in the two cases, since the two polynomials, considered as expressions in x and y, represent the same L, so that the coefficient of y in each case must be the same. We are thus led to an identity

$$\mathrm{A}_\alpha \mathrm{M} + \mathrm{B}_\alpha \mathrm{N} \equiv \mathrm{A}_\beta \mathrm{M}' + \mathrm{B}_\beta \mathrm{N},$$

in which A_α, B_α, A_β, B_β are known. A comparison of degrees shows that in general M, M′, and N are uniquely determined by this identity except for a constant factor ρ' of proportionality. Now this same L meets K again in what we have defined as $-P_{\alpha+\beta}$ or $P_{-(\alpha+\beta)}$. If $P_{\alpha+\beta}$ be defined by $\mathrm{A}_{\alpha+\beta} = 0$, $y = \mathrm{B}_{\alpha+\beta}$, then $-P_{\alpha+\beta}$ will be defined by using $(\mathrm{A}_{\alpha+\beta}, -\mathrm{B}_{\alpha+\beta})$. This same L may therefore be expressed by

$$\mathrm{A}_{\alpha+\beta} \mathrm{M}'' + (-\mathrm{B}_{\alpha+\beta})\mathrm{N} - y\mathrm{N}.$$

If $\mathrm{A}_{\alpha+\beta}$ were known, we could then solve for $-\mathrm{B}_{\alpha+\beta}$, and incidentally also for M″, by using the identity

$$\mathrm{A}_\alpha \mathrm{M} + \mathrm{B}_\alpha \mathrm{N} \equiv \mathrm{A}_{\alpha+\beta} \mathrm{M}'' - \mathrm{B}_{\alpha+\beta} \mathrm{N}.$$

But P_β, and $-P_{\alpha+\beta}$ are determined so far as their x-coördinates are concerned by a correspondence of the form (1). In fact, for the above determined M and N,

$$\mathrm{A}_\alpha \mathrm{M}^2 + 2\mathrm{B}_\alpha \mathrm{MN} + \Gamma_\alpha \mathrm{N}^2$$

vanishes for $A_\beta = 0$ and for $A_{\alpha+\beta} = 0$. Upon comparison of degrees, we see that

$$A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv \rho A_\beta A_{\alpha+\beta},$$

where ρ is a constant factor of proportionality. This serves to determine $A_{\alpha+\beta}$ which was all that was left to find. We shall suppose M and N so selected that $\rho^2 = 1$.

In brief, we obtain $A_{\alpha+\beta}$, and $B_{\alpha+\beta}$ from A_α, B_α, A_β, B_β, by the solution of a series of linear equations obtained by equating coefficients in the identities:

$$\begin{cases} A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv \rho A_\beta A_{\alpha+\beta}, \\ A_\alpha M + B_\alpha N \equiv A_\beta M' + B_\beta N \equiv A_{\alpha+\beta} M'' - B_{\alpha+\beta} N. \end{cases}$$

Explicit relations will appear in a more detailed form in § 12, and §§ 23–24.

If P_β coincides with P_α, the above method may be slightly simplified. In this case A_β is identical with A_α, so that $A_{2\alpha}$ is given directly, and incidentally also M and N, by

$$A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv \rho A_\alpha A_{2\alpha},$$

and $B_{2\alpha}$, by

$$A_\alpha M + B_\alpha N \equiv A_{2\alpha} M'' - B_{2\alpha} N.$$

A discussion of the cases in which this method may break down or be specialized owing to the vanishing of the determinant of the linear equations, will be made later. This applies to the following section also.

11. We shall now consider a shorter method of obtaining $A_{2\alpha}$ and $B_{2\alpha}$ than that just given. Let us determine a polynomial Λ in x such that

$$A_\alpha \Lambda + 2B_\alpha M + \Gamma_\alpha N \equiv 0,$$

which for a given A_α, B_α, Γ_α serves partially to determine Λ, M, N. We may completely determine them by requiring that the coefficient of the highest power of x in $M^2 - \Lambda N$ shall be equal to the above-mentioned ρ. The above bilinear form suggests two quadratic forms, whose discriminants are, respectively, $B_\alpha^2 - A_\alpha \Gamma_\alpha$ and $M^2 - \Lambda N$. The first of these is the Δ which we have used to define K, the second we shall now find to be simply $\rho A_{2\alpha}$. Indeed the identity

$$A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv \rho A_\alpha A_{2\alpha},$$

is found to be satisfied when $M^2 - \Lambda N$ is written instead of $\rho A_{2\alpha}$, for we have

$$A_\alpha M^2 + 2B_\alpha MN + \Gamma_\alpha N^2 \equiv A_\alpha M^2 - A_\alpha \Lambda N,$$

since $A_\alpha \Lambda \equiv -2B_\alpha M - \Gamma_\alpha N$. Hence for $\rho^2 = 1$,

$$A_{2\alpha} = \rho(M^2 - \Lambda N).$$

To obtain $B_{2\alpha}$, we must solve an identity of the form

$$A_\alpha M + B_\alpha N \equiv A_{2\alpha} M'' - B_{2\alpha} N,$$

that is,

$$A_\alpha M + B_\alpha N \equiv \rho(M^2 M'' - \Lambda N M'') - B_{2\alpha} N,$$

$$\equiv \rho(MM'' - N\Pi)M - (B_{2\alpha} + \rho\Lambda M'' - \rho M\Pi)N$$

where Π is arbitrary. If now we choose a Π and an M'' so that

$$\rho(MM'' - N\Pi) \equiv A_\alpha,$$

and that the coefficient of the highest power of x in $\rho(\Lambda M'' - M\Pi)$ is zero, the M'' and Π will be fully determined. We shall then have

$$A_\alpha M + B_\alpha N \equiv A_\alpha M - (B_{2\alpha} + \rho\Lambda M'' - \rho M\Pi)N,$$

whence

$$B_{2\alpha} \equiv -(B_\alpha + \rho\Lambda M'' - \rho M\Pi).$$

This method is a little more expeditious than the general method of determining $A_{2\alpha}$ and $B_{2\alpha}$ given in the preceding section owing to the fact that the polynomials whose coefficients we must equate are here of lower degree than by the method of the preceding section.

Explicit Conditions of Closure.

12. It will be more convenient for some purposes to use the following notation, which differs slightly from that used hitherto in this paper. On writing $-x_n$ for α_1 in the expression for $A_{n\alpha}$ (cf. § 8), y_n for β_1, and therefore $-(x_n^2 + \delta_1)$ for γ_1, we obtain

$$A_{n\alpha} \equiv x - x_n,$$

$$B_{n\alpha} \equiv y_n,$$

$$\Gamma_{n\alpha} \equiv -(x^2 + x_n x + x_n^2 + \delta_{1n}),$$

$$\Delta \equiv x^3 + \delta_1 x + \delta_2,$$

where $y_n = \pm\sqrt{x_n^3 + \delta_1 x_n + \delta_2}$. An equation $L = 0$, we may write in the general form:

$$y = l_0 x + l_1,$$

except in the cases when $N = 0$, in which cases we have

$$l_0 x + l_1 = 0.$$

If an L passes through $P_{m\alpha}$ and $P_{n\alpha}$, it will pass also through $-P_{(m+n)\alpha}$, while if one passes through $P_{m\alpha}$ and $P_{-n\alpha}$, it will pass also through $-P_{(m-n)\alpha}$, and by reflexion in the x-axis we obtain two other combinations. If we

be given only $A_{m\alpha}$ and $A_{n\alpha}$, it will be impossible from these data to determine whether $P_{m\alpha}$ or $-P_{m\alpha}$, and $P_{n\alpha}$ or $-P_{n\alpha}$ is intended. In fact $A_{m\alpha}$ and $A_{n\alpha}$ determine $A_{(m+n)\alpha}$ and $A_{(m-n)\alpha}$ by a quadratic construction. In other words, if we be given the x-coördinates only of two of the points of intersection of an L with K, the x-cöordinate of the remaining point of intersection is determined only as being one of a pair of uniquely determined points. We exclude for the moment the special case when the two given points have the same x-coördinate.

Eliminating y between

$$y^2 = x^3 + \delta_1 x + \delta_2, \qquad \text{and} \qquad y = l_0 x + l_1,$$

we obtain

$$x^3 - l_0^2 x^2 + (\delta_1 - 2l_0 l_1)x + (\delta_2 - l_1^2) = 0,$$

as the equation satisfied by the x-coördinates of the points of intersection. If we call these coördinates, x_m, x_n, and x' respectively we have the relations

$$x_m + x_n + x' = l_0^2,$$

$$x_m x_n + x_n x' + x' x_m - \delta_1 = -2l_0 l_1,$$

$$x_m x_n x' + \delta_2 = l_1^2.$$

Eliminating the l's, we have

$$(3) \quad (x_m x_n + x_n x' + x' x_m - \delta_1)^2 - 4(x_m + x_n + x')(x_m x_n x' + \delta_2) = 0.$$

This is a quadratic equation in x', whose roots we have called x_{m+n} and x_{m-n}. We may replace x' in this relation by $x'' + x_l$, where x_l is arbitrary, and obtain

$$(x_m x_n + x_n x'' + x_n x_l + x'' x_m + x_l x_m - \delta_1)^2$$
$$- 4(x_m + x_n + x'' + x_l)(x_m x_n x'' + x_m x_n x_l + \delta_2) = 0,$$

as a quadratic whose roots are $x_{m+n} - x_l$ and $x_{m-n} - x_l$. Expanding, we obtain

$$x''^2[(x_m - x_n)^2]$$
$$+ 2x''\,[(x_n + x_m)(x_m x_n + x_n x_l + x_l x_m - \delta_1)$$
$$- 2(x_m + x_n + x_l)x_m x_n - 2(x_m x_n x_l + \delta_2)]$$
$$+ [(x_m x_n + x_n x_l + x_l x_m - \delta_1)^2 - 4(x_m + x_n + x_l)(x_m x_n x_l + \delta_2)] = 0.$$

By taking the product of the roots we have

$$(x_m - x_n)^2(x_{m+n} - x_l)(x_{m-n} - x_l) = (x_m x_n + x_n x_l + x_l x_m - \delta_1)^2$$
$$- 4(x_m + x_n + x_l)(x_m x_n x_l + \delta_2).$$

By symmetry we have the same right-hand member for $(x_m - x_l)^2 (x_{m+l} - x_n)(x_{m-l} - x_n)$. Hence we have the relation

$$(x_m - x_n)^2(x_{m+n} - x_l)(x_{m-n} - x_l) = (x_m - x_l)^2(x_{m+l} - x_n)(x_{m-l} - x_n)$$

which holds for all choices of x_m, x_n, and x_l.

Let us now replace m and n in the above relation by ml and nl, and introduce expressions C having the following relation to the x's:

$$x_{ml} - x_l \equiv -\frac{C_{m+1}C_{m-1}}{C_m{}^2 C_1{}^2}.$$

The above formula now becomes

$$\left[\frac{C_{m+1}C_{m-1}}{C_m{}^2C_1{}^2} - \frac{C_{n+1}C_{n-1}}{C_n{}^2C_1{}^2}\right]^2\left[\frac{C_{m+n+1}C_{m+n-1}}{C^2_{m+n}C_1{}^2}\right]\left[\frac{C_{m-n+1}C_{m-n-1}}{C^2_{m-n}C_1{}^2}\right]$$

$$= \left[\frac{C_{m+1}C_{m-1}}{C_m{}^2C_1{}^2}\right]^2 \cdot \left[\frac{C_{m+2}C_m}{C^2_{m+1}C_1{}^2} - \frac{C_{n+1}C_{n-1}}{C_n{}^2C_1{}^2}\right] \cdot \left[\frac{C_mC_{m-2}}{C^2_{m-1}C_1{}^2} - \frac{C_{n+1}C_{n-1}}{C_n{}^2C_1{}^2}\right].$$

This may be written in the form:

$$\left[\frac{C_1{}^2C_{m+1+n}C_{m+1-n}}{C_{m+2}C_mC_n{}^2 - C_{n+1}C_{n-1}C^2_{m+1}}\right]\left[\frac{C_1{}^2C_{m-1+n}C_{m-1-n}}{C_mC_{m-2}C_n{}^2 - C_{n+1}C_{n-1}C^2_{m-1}}\right]$$

$$= \left[\frac{C_1{}^2C_{m+n}C_{m-n}}{C_{m+1}C_{m-1}C_n{}^2 - C_{n+1}C_{n-1}C_m{}^2}\right]^2.$$

We shall now define C_0 as equal to zero and C_{-m} as equal to $-C_m$. Then, for n arbitrary, and $m = 0$, or $m = 1$, we have identically

$$\frac{C_1{}^2C_{m+n}C_{m-n}}{C_{m+1}C_{m-1}C_n{}^2 - C_{n+1}C_{n-1}C_m{}^2} = 1.$$

But the preceding equation shows that if this equation holds for $m - 1$, and for m, then it must also continue to be true for $m + 1$. Therefore by mathematical induction, the C's are such that for all integer values of m and n

$$C_1{}^2C_{m+n}C_{m-n} = C_{m+1}C_{m-1}C_n{}^2 - C_{n+1}C_{n-1}C_m{}^2, \tag{4}$$

or, as we may write it,

$$\frac{C_{m+n}C_{m-n}}{C_m{}^2C_n{}^2} + \frac{C_{n+1}C_{n-1}}{C_n{}^2C_1{}^2} + \frac{C_{1+m}C_{1-m}}{C_1{}^2C_m{}^2} = 0. \tag{4'}$$

13. If we use the abbreviation $F_{m,\,n}$ for the fraction $C_{m+n}C_{m-n}/C_m{}^2C_n{}^2$, by we may derive immediately from (4'), that for any n_1, n_2 and n_3,

$$F_{n_1,\,n_2} + F_{n_2,\,n_3} + F_{n_3,\,n_1} = 0.$$

From the fact that by algebra we have directly

$$(F_{n_1,\,m} - F_{n_2,\,m})(F_{n_3,\,m} - F_{n_4,\,m}) + (F_{n_2,\,m} - F_{n_3,\,m})(F_{n_1,\,m} - F_{n_4,\,m}) + (F_{n_3,\,m} - F_{n_1,\,m})(F_{n_2,\,m} - F_{n_4,\,m}) \equiv 0,$$

we conclude that

$$F_{n_1,\,n_2} \cdot F_{n_3,\,n_4} + F_{n_2,\,n_3} \cdot F_{n_1,\,n_4} + F_{n_3,\,n_1} \cdot F_{n\,,\,n_4} = 0,$$

which gives in terms of the C's

$$C_{n_1+n_2}C_{n_1-n_2}C_{n_3+n_4}C_{n_3-n_4} + C_{n_2+n_3}C_{n_2-n_3}C_{n_1+n_4}C_{n_1-n_4} + C_{n_3+n_1}C_{n_3-n_1}C_{n_2+n_4}C_{n_2-n_4} = 0.$$

Identities such as these serve frequently to greatly abridge the work of determining a particular C in terms of other C's. We shall now give however a general method of determining each C in terms of the preceding C's. In fact we merely need to write, in (4), first, $n + 1$ in place of m, and obtain

$$C_{2n+1} = \frac{1}{C_1^{\,3}}[C_{n+2}C_n^{\,3} - C_{n-1}C_{n+1}^3], \tag{5}$$

and secondly, $n + 1$ in place of m, and $n - 1$ in place of n, and obtain

$$C_{2n} = \frac{C_n}{C_1^{\,2}C_2}[C_{n+2}C_{n-1}^2 - C_{n-2}C_{n+1}^2]. \tag{5'}$$

These formulæ (5) and (5′) serve to determine every C_n, if C_1, C_2, C_3, C_4, be known. Furthermore if C_2, C_3, and C_4, be divisible by C_1, formulæ (5) and (5′) serve to establish by induction that every suceeding C is also divisible by C_1. Also, if C_4 is divisible by C_2, then for n odd, or even, every C_{2n} must be divisible by C_2. For suppose this to be the case for $n \leqq \nu + 2$. If n be odd, C_{n-1}^2 and C_{n+1}^2 will both contain $C_2^{\,2}$ as a factor so that from formula (5′), C_{2n} will contain C_2 exactly. If n be even, C_n as well as both C_{n+2}, and C_{n-2} will contain C_2, so that again C_{2n} contains C_2 as a factor. Hence, *any choice of* $C_1, C_2, C_3,$ *and* C_4, *for which* $C_1, C_2/C_1, C_3/C_1,$ *and* $C_4/(C_1 \times C_2)$, *are all polynomials in* $x_l, y_l, \delta_1, \delta_2$, *will be such that every* C_n *will be a polynomial in these same quantities*, $x_l, y_l, \delta_1, \delta_2$.

14. We may notice that in (4) not only are the terms homogeneous, each being of the same (fourth) degree in the C's, but there is also a homogeneity of another kind, namely with respect to the *square* of the subscripts. Indeed

$$2 \cdot 1^2 + (m + n)^2 + (m - n)^2 = (m + 1)^2 + (m - 1)^2 + 2n^2$$
$$= (n + 1)^2 + (n - 1)^2 + 2m^2 = 2(1^2 + m^2 + n^2),$$

which we may call the " quadratic weight " of the expressions involved. Furthermore any expression derived from (4) will be homogeneous in both senses. It will be possible to form from C_1, C_2 and C_n, an expression of degree zero and quadratic weight zero, which shall reduce to unity for $n = 1$, and $n = 2$, and vanish for $n = 0$. Indeed we may write

$$R_n \equiv \frac{C_n \cdot C_1^{(n^2-4)/3}}{C_2^{(n^2-1)/3}},$$

where R_n is a function of the form desired. For every identical relation among the C's we have an analogous relation among the R's, obtainable by merely replacing C_n by R_n. In particular we have formulæ analogous to (5) and (5′), where in the present case we may replace R_1 and R_2 by their common value, unity. Thus we have

$$R_{2n+1} = R_{n+2}R_n^3 - R_{n-1}R_{n+1}^3, \tag{6}$$

$$R_{2n} = R_n(R_{n+2}R_{n-1}^2 - R_{n-2}R_{n+1}^2). \tag{6′}$$

It will be found that R_3 occurs throughout the entire series of expressions R_n only in the form R_3^3, except in those R_n's for which n is a multiple of three, in which case R_3 occurs also as a simple factor of the entire expression, as is readily proved by induction.

15. We may write U for R_3^3 and V for R_4. We then have directly from (6) and (6′),

$$R_0 = 0,$$

$$R_1 = 1,$$

$$R_2 = 1,$$

$$R_3 = U^{1/3},$$

$$R_4 = V,$$

$$R_5 = V - U,$$

$$R_6 = U^{1/3}(V - U - V^2),$$

$$R_7 = (V - U)U - V^3,$$

$$R_8 = V[(V - U)(2U - V) - UV^2],$$

$$R_9 = U^{1/3}[V^3(V - U - V^2) - (V - U)^3],$$

$$R_{10} = (V - U)[V^2(UV - U^2 - V^3) - U(V - U - V^2)^2],$$

$$R_{11} = (UV - U^2 - V^3)(V - U)^3 - UV(V - U - V^2)^3,$$

.

$$R_{-n} = -R_n.$$

16. Thus far, the C's and the R's have depended upon an arbitrary integer l. Putting $l = 1$, we have

$$x_m - x_1 = -\frac{C_{m+1}C_{m-1}}{C_m{}^2C_1{}^2},$$

and more generally from (4), we have

$$x_m - x_n = -\frac{C_{m+n}C_{m-n}}{C_m{}^2C_n{}^2}.$$

Now x_0 is the x at infinity so that $x_m - x_n$ becomes infinite when and only when either x_m or x_n coincides with x_0. Furthermore $x_m - x_n$ vanishes when and only when either x_{m+n} or x_{m-n} coincides with x_0. Thus the vanishing of C_m is a necessary and sufficient condition that x_m coincides with x_0, provided only that the C's be integral functions of x_1 and y_1. This may be readily secured. We have merely to take C_1 equal to unity, and C_2 equal to $-2y_1$, although other choices are also possible. We will find immediately that C_3 then assumes an integral form, and as we shall show in the following section, § 17, C_4 is then integral and divisible by C_2. The discussion in § 13 shows that all the C's are integral functions of x_1 and y_1.

The identity $\mathrm{A}_a\Lambda + 2\mathrm{B}_a\mathrm{M} + \Gamma_a\mathrm{N} \equiv 0$, admits, when ρ is taken as minus one, the solutions*

$$\Lambda \equiv x + 2x_1,$$

$$\mathrm{M} \equiv \frac{1}{2y_1}(3x_1{}^2 + \delta_1),$$

$$\mathrm{N} \equiv 1.$$

The identity

$$\rho(\mathrm{MM}'' - \mathrm{N}\Pi) \equiv \mathrm{A}_a,$$

yields, cf. § 11

$$\mathrm{M}'' \equiv \frac{1}{2y_1}(3x_1{}^2 + \delta_1),$$

$$\Pi \equiv (x - x_1) + \frac{1}{4y_1{}^2}(3x_1{}^2 + \delta_1)^2.$$

The equation, $\rho(\mathrm{M}^2 - \Lambda\mathrm{N}) = A_{2a}$, gives

$$A_{2a} = x + 2x_1 - \frac{1}{4y_1{}^2}(3x_1{}^2 + \delta_1)^2,$$

$$= x - \frac{1}{4y_1{}^2}(x_1{}^4 - 2\delta_1x_1{}^2 - 8\delta_2x_1 + \delta_1{}^2).$$

* It may be seen that we must throughout suppose that every N is taken as unity, that ρ is always minus unity, and M, M′, and M″ for a given L always coincide, in order that the normalizing conditions which we have imposed be satisfied; but cf. §§ 23–24.

Similarly the equation $B_{2\alpha} = -(B_\alpha + \rho\Lambda M'' - \rho M\Pi)$ gives

$$B_{2\alpha} = \frac{1}{8y_1^3}[x_1^6 + 5\delta_1 x_1^4 + 20\delta_2 x_1^2 - 5\delta_1^2 x_1^2 - 4\delta_1\delta_2 x_1 - (8\delta_2^2 + \delta_1^3)].$$

Since $A_{2\alpha} = x - x_2$, we may compute C_3 from

$$x_2 - x_1 = -\frac{C_3 C_1}{C_2^2 C_1^2}.$$

In fact

$$C_3 = 3x_1^4 + 6\delta_1 x_1^2 + 12\delta_2 x_1 - \delta_1^2.$$

We shall now determine C_4.

17. From the quadratic (3), whose roots are x_{m+n} and x_{m-n}, we obtain for the sum of the roots:

$$(7) \qquad x_{m+n} + x_{m-n} = \frac{2}{(x_m - x_n)^2}[(x_m x_n + \delta_1)(x_m + x_n) + 2\delta_2].$$

Now $(x_{m-n} - x_1)(x_m - x_n)^2$ may be written in the form

$$-\frac{C_{m-n+1}C_{m-n-1}}{C_{m-n}^2 C_1^2} \cdot \frac{C_{m+n}^2 C_{m-n}^2}{C_m^4 C_n^4} = \frac{C_{m-n+1}C_{n-m+1}C_{m+n}^2}{C_m^4 C_n^4}.$$

Thus, although $(x_m - x_n)^2$ vanishes for $n = m$, yet the expression $(x_{m-n} - x_1)(x_m - x_n)^2$ reduces for $n = m$ to C_{2m}/C_m^8. Hence for $n = m$, we obtain from (7)

$$\begin{aligned} C_{2m} &= 2C_m^8[(x_m^2 + \delta_1)2x_m + 2\delta_2], \\ &= 4C_m^8 \cdot y_m^2, \\ &= 4C_m^8 \cdot B_{m\alpha}^2. \end{aligned}$$

The sign of $B_{m\alpha}$ depends upon the sign chosen for B_α, i. e., y_1, and may be found from the identities

$$A_\alpha M + B_\alpha \equiv A_{m\alpha} M + B_{m\alpha} \equiv A_{(m+1)\alpha} M - B_{(m+1)\alpha}.$$

Eliminating M, this gives

$$[A_{(m+1)\alpha} - A_\alpha][B_{m\alpha} - B_\alpha] + [A_{m\alpha} - A_\alpha][B_{(m+1)\alpha} + B_\alpha] \equiv 0,$$

which is satisfied identically for $B_{m\alpha} = -(C_{2m}/2C_m^4)$, and $B_{(m+1)\alpha} = -(C_{2(m+1)}/2C_{m+1}^4)$, but not for any other combinations of signs.

Since, now, we have already found $B_{2\alpha}$, we may determine C_4. Indeed, we have immediately,

$$C_4 = -4y_1[x_1^6 + 5\delta_1 x_1^4 + 20\delta_2 x_1^2 - 5\delta_1^2 x_1^2 - 4\delta_1\delta_2 x_1 - (8\delta_2^2 + \delta_1^3)].$$

Having found C_2, C_3 and C_4, we may either find the R's first, or compute

the C's directly by the recursion formulæ, (5) and (5′). The C's determine a set of "conditions of closure," $C_n = 0$ being a "condition of closure of the nth order."

Thus *every* C_n *is a polynomial in* x_1; *for* n *odd the coefficients of the various powers of* x_1, *are themselves polynomials in* δ_1 *and* δ_2 *with integral numerical coefficients*; *for* n *even, the same is true except that* y_1 *also occurs as a factor of the entire expression.* We have just seen this to be the case for C_2, C_3 and C_4. Formulæ (5) and (5′) serve immediately to establish by induction the truth of this statement, in all cases.

Generalization to Hyperelliptic Curves.

18. The statements in §§ 3–6, 8–11, inclusive, are capable of a wide extension without even verbal modification. We shall now proceed to present this more general problem.

Let us take an arbitrary positive integer p, and form a polynomial $\Delta(x)$ of degree $2p + 1$, of the form,

$$\Delta(x) \equiv x^{2p+1} + \delta_1 x^{2p-1} + \delta_2 x^{2p-2} + \cdots + \delta_{2p-1} x + \delta_{2p},$$

the δ's being constants. The equation $y^2 = \Delta(x)$ defines a curve K, which is said to be hyperelliptic and of genus p. Furthermore this equation may be considered as a normal form under a certain class of transformations, viz., the birational group, of the most general non-degenerate hyperelliptic curve. We shall suppose that $\Delta(x) = 0$ does not admit multiple roots, as otherwise the curve has finite multiple points, which we shall suppose to have been removed. We shall carry over the definition given in § 3, of "general position," extending it only by defining a set some of whose points are at infinity as being in general position, if the set of the finite points when existent is in general position in the sense of the previous definition. An arbitrary set of p points in general position on K, shall be defined as a set P or simply as a P.

There are certain variable curves L, having for equations, expressions of the form $L(x, y) = 0$, which meet K in $3p$ variable points of intersection, with perhaps one common fixed point at infinity. These curves are such that any $2p$ of the $3p$ variable points of intersection may be assigned at random, but the curve L is thereby completely identified, and the positions of the remaining p intersections are then determinate. We shall consider the form of L, and show that the L's which we obtain satisfy not only the properties (a) and (b), but also (c) and (d), mentioned in § 3 as properties of the cubic. Incidentally we shall be proving the theorems in that case also.

19. The equation of an algebraic curve F, in the (x, y) plane can be written in the form $F(x, y) = 0$, where $F(x, y)$ is a polynomial in x and y. We may separate $F(x, y)$ into an even and an odd function of y so as to write

$$F(x, y) \equiv F_0(x, y^2) + yF_1(x, y^2).$$

The curve F, in so far as the equation $y^2 = \Delta(x)$ is supposed to be satisfied, i. e., in so far as the points of K alone are considered, is the same as the curve F', whose equation is

$$F'(x, y) \equiv F_0[x, \Delta(x)] + yF_1[x, \Delta(x)].$$

Clearly this is finite for all finite points on the curve K. Moreover, every rational function of x and y, which is finite for all finite points on the curve K, may be written in the form, $F_0(x) + yF_1(x)$.

For suppose we be given any rational function $G(x, y)/H(x, y)$, where G and H are polynomials. If we write H in the form $H_0(x, y^2) + yH_1(x, y^2)$, and then multiply numerator and denominator of the fraction by $H_0 - yH_1$, the denominator will become an even function of y. Replacing y^2 throughout by $\Delta(x)$, we shall have an expression of the form

$$\frac{F_0(x) + yF_1(x)}{H_2(x)}.$$

We wish now to show that if this is to be finite for all finite points on the curve, then the denominator $H_2(x)$ is a factor of both $F_0(x)$ and $F_1(x)$.

If $[F_0(x) + yF_1(x)]/H_2(x)$ is to be finite, the same is true of $[F_0(x) - yF_1(x)]/H_2(x)$, for these differ only in the sign of y, while the curve $y^2 = \Delta(x)$ is symmetric with respect to the x-axis, so that should one become infinite at a point (x, y) the other must become infinite at $(x, -y)$. Both the sum and the product of finite functions are finite so that

$$\frac{2F_0(x)}{H_2(x)} \quad \text{and} \quad \frac{F_0^2(x) - y^2F_1^2(x)}{H_2^2(x)} \quad \text{or} \quad \frac{F_0^2(x) - \Delta(x)F_1^2(x)}{H_2^2(x)},$$

are finite for finite points on K. From the first expression we see that every root of the polynomial $H_2(x)$ must also be a root of the polynomial $F_0(x)$, and hence by the second expression, every simple root of $H_2(x)$ is a double root of ΔF_1^2. But by hypothesis $\Delta(x) = 0$ has no double roots, so that any root of $H_2(x)$, is also a root of $F_1(x)$. Hence every factor of $H_2(x)$ is a factor of both $F_0(x)$ and of $F_1(x)$, and we may divide out $H_2(x)$ entirely, as was to be proved.

20. If either $F_0(x)$ or $F_1(x)$ be identically zero or, more generally, if F_0 and F_1 be not prime to each other, then the intersections of the curve

$F \equiv F_0(x) + yF_1(x) = 0$, with the curve K, will not be in general position, since every common linear factor of F_0 and F_1 determines two points on K, which differ only in the sign of y. Let us suppose F_0 and F_1 prime to each other. The x-coördinates of the points of intersection of $F \equiv F_0(x) + yF_1(x) = 0$, with $y^2 = \Delta(x)$ are given by the roots of the equation

$$F_0^2(x) - \Delta(x)F_1^2(x) = 0.$$

Since $\Delta(x)$ is of *odd* degree, and $F_0^2(x)$ and $F_1^2(x)$ are necessarily of *even* degree, it will be impossible for the leading coefficients of $F_0^2(x)$ and $\Delta(x) \cdot F_1^2(x)$ to cancel each other. Let us suppose r, $\geqq p + 1$, points of intersection chosen arbitrarily and in general position on K. This will impose r linear conditions upon the coefficients of $F \equiv F_0 + yF_1 = 0$. For $r - p$ even, we shall write $r = p + 2s$, and for $r - p$ odd, we shall write $r = p + 2s - 1$, where $s \geqq 1$ in both cases. The highest possible degrees of F_0, and of F_1, for which $F_0^2 - \Delta F_1^2$ shall be of degree $r + p$ are indicated by the following table to be $r - s$ and $s - 1$, respectively.

Terms	*Degree*	*Deg. for* $r - p = 2s - 1$	*Deg. for* $r - p = 2s$
F_0	$r - s$	$p + s - 1$	$p + s$
F_1	$s - 1$	$s - 1$	$s - 1$
F_0^2	$2r - 2s$	$r + p - 1$	$r + p$
ΔF_1^2	$2p + 2s - 1$	$r + p$	$r + p - 1$
$F_0^2 - \Delta F_1^2$	$r + p$	$r + p$	$r + p$

Now the total number of homogeneous coefficients in F_0 is $r - s + 1$, and in F_1 is s, so that if Δ be given and fixed, $F \equiv F_0 + yF_1$ can be subjected to exactly r linearly independent conditions, while, on the other hand, the number of intersections of F with K will be $r + p$. Hence, the curve F which has $r + p$ intersections with K cannot have more than r independent coefficients, it being supposed that $r \geqq p + 1$. The statement also holds for $r = p$ since $F_0^2 - \Delta F_1^2$ can be of degree $2p$ only when F_1 vanishes identically, Δ being of degree $2p + 1$, since, as we have remarked, the leading coefficients of the two terms can never cancel each other. But in this case, F_0 must be of degree p exactly, and the equation $F_0^2 = 0$ contains but p independent coefficients.

21. We may choose as the curves L, the curves whose equations are of the form $L(x, y) \equiv E(x) - y\mathrm{N}(x) = 0$, where E and $-\mathrm{N}$ are polynomials of degree $2p - q$ and $q - 1$, respectively, q being $\frac{1}{2}p$ when p is even, and $\frac{1}{2}(p + 1)$ when p is odd. Thus E and $-N$ are cases of F_0 and F_1, for which $r = 2p$ and $s = q$. Hence if $2p$ points, i. e., two P's, be given on K, no two of the points having the same x-coördinate, then there is a unique L through them. Since $E^2 - \Delta\mathrm{N}^2$ is of degree $3p$, it will vanish

for yet a third P the y-coördinate of each point of which is determined linearly by the equation $\mathrm{N}y = E$. Thus property (a) holds true for the general hyperelliptic curve, K.

The condition that the $3p$ points of intersection of an L with K, reduce to p sets of three coincident points each, i. e., that $E^2 - \Delta\mathrm{N}^2$ be a perfect cube, clearly requires that the coefficients of E and N satisfy certain algebraic relations. Indeed if we write

$$E^2 - \Delta\mathrm{N}^2 \equiv Q^3,$$

where Q is polynomial in α of degree p all of whose coefficients are unknowns, we shall have $3p + 1$ equations which are homogeneous in the $2p + 1$ homogeneous coefficients of E and N, and of degree two in these, and also homogeneous and of degree three in the $p + 1$ homogeneous coefficients of Q. The number of solutions will be found to be 3^{2p}. Thus property (b) holds also in this general case.

22. We shall now make the hypothesis required for property (c), as stated in § 3. The curves F' and F'' of § 3, may be represented by equations of the forms $F_0'(x) + yF_1'(x) = 0$ and $F_0''(x) + yF_1''(x) = 0$ respectively, so far as points of K are concerned. Now these two rational functions are known to have r of their total $r + p$ zeros on K in common and in general position on K. But these r zeros which constitute the set S fully determine the rational function, except for a constant factor. Thus $F_0'(x) + yF_1'(x)$ and $F_0''(x) + yF_1''(x)$ differ from each other by at most a constant factor, and have all of their zeros on K in common. This proves (c) to hold true for the hyperelliptic case.

In order that $E^2(x) - \Delta(x)\mathrm{N}^2(x) = 0$, shall have p coincident roots at infinity, it is necessary and sufficient that N vanish identically, while E reduces to degree p. In this case L reduces to p straight lines parallel to the y-axis. In the equations satisfied by an inflexional set of p points, we do not use y explicitly and N enters in these, only in the form N^2. Hence if $E - \mathrm{N}y$ yields one solution, $E + \mathrm{N}y$ will give another. Thus (d) also is valid in the present problem. The entire discussion in §§ 3–6, inclusive, applies to the hyperelliptic case, since it follows from the properties (a), (b), (c), (d).

23. We may now start with the correspondence

$$\mathrm{A}(x)\mathrm{M}^2(x) + 2\mathrm{B}(x)\mathrm{M}(x)\mathrm{N}(x) + \Gamma(x)\mathrm{N}^2(x) = 0, \tag{1}$$

where A, B, Γ, M, N, are polynomials in x. We may suppose the degrees of A, B and Γ to be given as equal to $p + 1$. By applying the reductions used in § 8, we find that, finally, if our M and N have been suitably chosen, A, B, Γ, M, N are of the following form.

$$\begin{aligned}
\mathrm{A} &\equiv x^p + \alpha_1 x^{p-1} + \alpha_2 x^{p-2} + \cdots + \alpha_p, \\
\mathrm{B} &\equiv \beta_1 x^{p-1} + \beta_2 x^{p-2} + \cdots + \beta_p, \\
\Gamma &\equiv -x^{p+1} + \alpha_1 x^p + \gamma_1 x^{p-1} + \gamma_2 x^{p-2} + \cdots + \gamma_p, \\
\mathrm{M} &\equiv \mu_0 x^{p-q} + \mu_1 x^{p-q-1} + \cdots + \mu_{p-q}, \\
\mathrm{N} &\equiv \nu_0 x^{q-1} + \nu_1 x^{q-2} + \cdots + \nu_{q-1},
\end{aligned}$$

and therefore where $\Delta \equiv \mathrm{B}^2 - \mathrm{A}\Gamma$ is of the form

$$\Delta \equiv x^{2p+1} + \delta_1 x^{2p-1} + \delta_2 x^{2p-2} + \cdots + \delta_{2p}.$$

The E that we have previously used may hereafter be considered as having been an abbreviation for $\mathrm{AM} + \mathrm{BN}$. We have merely to take the x-coördinates of p of the points of intersection of $E(x) - y\mathrm{N}(x)$ with K, and form the equation $\mathrm{A}(x) = 0$ which has these for roots. The y-coördinates of these same p points may be used to determine a corresponding B by means of the equations

$$y_i = \beta_1 x_i^{p-1} + \beta_2 x_i^{p-2} + \cdots + \beta_p \qquad i = 1, 2, \cdots p,$$

whose determinant is the Vandermonde determinant of the x's. Since the p points (x_i, y_i) are supposed to have their x_i-coördinates distinct, we may always determine the desired $\mathrm{B}(x)$, and determine it uniquely. But this choice of A and B satisfies the condition that for $\mathrm{A} = 0$, E reduces to NB. If we let the x-coördinates of the remaining points of intersection of $y = \mathrm{B}$ with L, be the roots of a polynomial M put equal to zero, the E will be necessarily of the form, $E \equiv \mathrm{BN} + \mathrm{AM}$ if the arbitrary constant factor in N and M be properly chosen.

The remarks of §§ 8–10, inclusive, are now immeditely applicable to the correspondence (1), extended in this manner.

It may be remarked that when working in the projective plane, i. e., closing the finite plane by means of a line at infinity, the point at infinity on the curve is by no means a simple, but is a p-tuple point. For most purposes, it is convenient to suppose K situated in a "function plane." In other words, we shall in speaking of the point at infinity regard A, B, Γ as *forms*, which do not become infinite at infinity, while, on the other hand, A and B vanish there to the first and second orders, respectively. With this convention, infinity becomes an ordinary branch point for the curve K, and the L's do not in general pass through this point. This matter is not however of great significance in this connection, as we are concerned almost exclusively with those points alone which lie on the curve K, whose internal properties are essentially the same in either plane.

24. In the identity

$$\mathrm{A}\Lambda + 2\mathrm{BM} + \Gamma\mathrm{N} \equiv 0,$$

the term ΓN is of degree $p + q$, which is always higher than the degree $2p - q - 1$ of the term 2BM. But A is itself of degree p, so that Λ must be of degree q, since its leading term must cancel that of ΓN. So that we must have $\lambda_0 = \nu_0$, where λ_0 is the leading coefficient in Λ. Now Λ, M, N contain altogether $(q + 1) + (p - q + 1) + q = p + q + 2$ homogeneous coefficients and since the degree of the left hand member is $p + q$, we have $p + q + 1$ homogeneous equations to be satisfied. Thus the solution is unique apart from an arbitrary factor. In general we may impose the condition that for $p = 2q - 1$, $\lambda_0 = 1$, and for $p = 2q$, $\mu_0 = 1$. By this means we shall secure that the leading coefficient in $M^2 - \Lambda N$ is equal to $\rho = (-1)^p$. The Λ, M, N will now be fully determined if the A, B, Γ are supposed given.

In the identity

$$\rho(MM'' - N\Pi) \equiv A$$

where M'' is of degree $p - q$, and Π of degree q, the coefficient of x^p in the left-hand member is, for $p = 2q - 1$, π_0, and for $p = 2q$, μ_0''. And since this is to be an identity, we must have for $p = 2q - 1$, $\pi_0 = 1$, and for $p = 2q$, $\mu_0'' = 1$. The degree of the left hand member is p, while the number of coefficients in M'' and Π together is $p + 2$. Since the equations which are obtained by equating coefficients are non-homogeneous and only $p + 1$ in number, we may in general impose an additional linear condition. We shall require for $p = 2q - 1$, $\mu_0'' = \mu_0$, and for $p = 2q$, $\pi_0 = \lambda_0$. With this restriction, the M'' and Π are completely determined, and in such a manner that $\lambda_0\mu_0'' - \pi_0\mu_0 = 0$, for p odd or even, so that in either case the coefficient of x^p in $\rho(\Lambda M'' - \Pi M)$ is zero. The discussion in § 11, now applies without modification to the hyperelliptic case.

It is important to note that in the above nonhomogeneous system the determinant of the coefficients of the system is for $p = 2q - 1$ and also for $p = 2q$, precisely the resultant of M and N. But if M and N have a common factor, the L whose equation is $AM + BN - Ny = 0$, and which has contact at the P defined by the (A, B) in question, will pass through the point at infinity, and will contain one or more lines parallel to the y-axis. But these lines can have contact with the curve only at the branch points. Thus the above method breaks down only when the given set P has one or more points at the branch points. But the situation in this special case is too obvious to require further discussion.

The General Algebraic Curve.

25. On any algebraic curve of genus p, we may consider an arbitrary series of groups g_{3p}^{2p} of points, any group of which contains $3p$ points and is determined by $2p$ of them. There will be, of course, in general, certain

fixed points in addition, should we cut the series out by a linear family of adjoints. Since $3p > 2p - 2$, the series cannot be special. There exist certain " inflexional " groups G of p sets of three coincident points each. We may choose one such set as P_0, and then determine with respect to this P_0, an arithmetic of groups of p points on the curve. It is only in the hyperelliptic case, however, that an individual point can be considered as having a negative obtained by a " reflexion." The Riemann-Roch Theorem in its simplest form shows that the property expressed in (c) holds good for the general algebraic curve. Thus there is an analogous theorem of closure for any algebraic curve of genus $p > 0$.

The Transcendental Treatment.

26. The Inversion Problem of Jacobi is concerned with the correspondence between the coördinates of a set of p points on an algebraic curve of genus p and the p sums of p linearly independent integrals of the first kind, the sum in each case being of the values assumed by one of these integrals at each of the p points in question. If we consider a group P to represent the p sums here mentioned, then by the addition of P's is meant only the addition of integrals, so that the associative law no longer needs proof, when once it is proved that the group of points is uniquely determined by the p sums of integrals. The equations now become linear equations, *modulis* the periods.

If in the elliptic case we take $x_1 = \wp(u)$ where $\wp$ is the Weierstrassian function, then x_n is $\wp(nu)$, and C_n is a theta function of u of the nth order most conveniently written in the form

$$C_n = \frac{\sigma(nu)}{\sigma(u)^{n^2}}.$$

The algebraic treatment does not define C_n except for n an integer, and by reversing the processes, for n, a fraction. It may be noted, however, that the C_n's are defined for curves in modular geometries by the present treatment, where transcendental methods cannot be applied.

The author wishes to express his thanks to Professor Oswald Veblen for numerous suggestions and corrections which have modified the form of this paper.

PRINCETON, N. J.,
November 12, 1914.

AN INTEGRAL EQUATION OF THE VOLTERRA TYPE.*

By T. H. Gronwall.

1. Consider the integral equation

$$\varphi(x, y) = f(x, y) + \int_0^x A(x, y, s)\varphi(s, y)ds + \int_0^y B(x, y, t)\varphi(x, t)dt + \int_0^x ds \int_0^y C(x, y, s, t)\varphi(s, t)dt, \tag{1}$$

where the given functions f, A, B and C are bounded and have their discontinuities regularly distributed in the region

$$0 \leqq s \leqq x \leqq a, \qquad 0 \leqq t \leqq y \leqq b,$$

or briefly, in the rectangle (a, b). Let the constants F and M be chosen so that, in the rectangle (a, b),

$$|f(x, y)| \leqq F, \quad |A(x, y, s)| \leqq M, \quad |B(x, y, t)| \leqq M, \quad |C(x, y, s, t)| \leqq M^2. \tag{2}$$

The usual method† of showing that there exists one and only one solution $\varphi(x, y)$ of (1) which is bounded in the rectangle (a, b), consists in subdividing this rectangle into smaller ones with sides α and β parallel to the x- and y-axis respectively; if α and β satisfy the condition

$$M\alpha + M\beta + M^2\alpha\beta < 1,$$

the method of successive approximations converges in the rectangle (α, β), and the solution obtained is then successively extended to adjacent rectangles by a process of continuation. It is the purpose of the present note to establish directly the existence and uniqueness of the bounded solution $\varphi(x, y)$ in the entire rectangle (a, b).

Introducing an arbitrary parameter λ, we replace equation (1) by

$$\varphi(x, y) = f(x, y) + \lambda\int_0^x A(x, y, s)\varphi(s, y)ds + \lambda\int_0^y B(x, y, t)\varphi(x, t)dt + \lambda^2\int_0^x ds \int_0^y C(x, y, s, t)\varphi(s, t)dt, \tag{3}$$

which reduces to (1) for $\lambda = 1$.‡

* Read before the American Mathematical Society, Jan. 1, 1915.

† See for instance, T. Lalesco, Introduction à la théorie des équations intégrales (Paris, Hermann, 1912) pp. 14–15.

‡ In the usual method, the double integral is multiplied by λ instead of λ^2.

Assuming for $\varphi(x, y)$ a uniformly convergent expansion

$$\varphi(x, y) = \sum_{n=0}^{\infty} \varphi_n(x, y)\lambda^n, \tag{4}$$

introducing this in (3) and comparing coefficients of equal powers of λ, we obtain

$$\begin{aligned}
\varphi_0(x, y) &= f(x, y), \\
\varphi_1(x, y) &= \int_0^x A(x, y, s)\varphi_0(s, y)ds + \int_0^y B(x, y, t)\varphi_0(x, t)dt, \\
\varphi_n(x, y) &= \int_0^x A(x, y, s)\varphi_{n-1}(s, y)ds + \int_0^y B(x, y, t)\varphi_{n-1}(x, t)dt \\
&\quad + \int_0^x ds \int_0^y C(x, y, s, t)\varphi_{n-2}(s, t)dt, \qquad (n \geqq 2).
\end{aligned} \tag{5}$$

From (2) and (5) we obtain

$$|\varphi_0(x, y)| \leqq F,$$

$$|\varphi_1(x, y)| \leqq FM\int_0^x ds + FM\int_0^y dt \leqq \frac{F \cdot 3M(x + y)}{1!},$$

so that the relation

$$|\varphi_n(x, y)| \leqq \frac{F \cdot (3M)^n(x + y)^n}{n!} \tag{6}$$

holds for $n = 0$ and $n = 1$. Supposing (6) proved for values of the subscript up to $n - 1$, the last equation in (5) gives

$$|\varphi_n(x, y)| \leqq \frac{F \cdot 3^{n-1}M^n}{(n-1)!}\left[\int_0^x (s + y)^{n-1}ds + \int_0^y (x + t)^{n-1}dt\right] + \frac{F \cdot 3^{n-2}M^n}{(n-2)!}\int_0^x ds \int_0^y (s + t)^{n-2}dt,$$

and from the obvious inequalities

$$\int_0^x (s + y)^{n-1}ds = \frac{(x + y)^n - y^n}{n} \leqq \frac{(x + y)^n}{n},$$

$$\int_0^y (x + t)^{n-1}dt = \frac{(x + y)^n - x^n}{n} \leqq \frac{(x + y)^n}{n},$$

$$\int_0^x ds \int_0^y (s + t)^{n-2}dt \leqq \int_0^x \frac{(s + y)^{n-1}}{n - 1}ds \leqq \frac{(x + y)^n}{n(n - 1)},$$

it follows that

$$|\varphi_n(x, y)| \leqq \frac{F \cdot (3M)^n(x + y)^n}{n!}\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{9}\right) \leqq \frac{F \cdot (3M)^n(x + y)^n}{n!},$$

so that (6) is true for every value of n. We conclude from (6) that in the rectangle (a, b)

$$| \varphi_n(x, y) | \leqq \frac{F \cdot (3M)^n(a + b)^n}{n!},$$

so that the series (4) is uniformly convergent in (a, b). Making $\lambda = 1$, we obtain a solution of (1) which is bounded in (a, b).

2. To prove the uniqueness of this bounded solution $\varphi(x, y)$, suppose $\varphi_1(x, y)$ to be another. The difference $u(x, y) = \varphi_1(x, y) - \varphi(x, y)$ is then also bounded in the rectangle (a, b):

$$| u(x, y) | < \mu, \tag{7}$$

and satisfies the integral equation

$$u(x, y) = \int_0^x A(x, y, s)u(s, y)ds + \int_0^y B(x, y, t)u(x, t)dt + \int_0^x ds \int_0^y C(x, y, s, t)u(s, t)dt. \tag{8}$$

Consequently

$$| u(x, y) | \leqq \mu M \int_0^x ds + \mu M \int_0^y dt + \mu M^2 \int_0^x ds \int_0^y dt = \mu(Mx + My + M^2xy)$$

$$\leqq \mu(Mx + My + M^2ay) \leqq \mu(M + M^2a)(x + y),$$

or writing $K = M + M^2a$, so that $K > M$,

$$| u(x, y) | \leqq \mu K(x + y) \leqq \frac{\mu \cdot 3K(x + y)}{1!},$$

so that the relation

$$| u(x, y) | \leqq \frac{\mu \cdot (3K)^m(x + y)^m}{m!} \tag{9}$$

holds for $m = 0$ and $m = 1$. Suppose this relation to be true for $m \leqq n-1$, we obtain from (8), using (9) with $m = n - 1$ in the single and $m = n-2$ in the double integral,

$$| u(x, y) | \leqq \mu \left[\frac{M \cdot (3K)^{n-1}(x + y)^n}{n!} + \frac{M \cdot (3K)^{n-1}(x + y)^n}{n!} + \frac{M^2 \cdot (3K)^{n-2}(x + y)^n}{n!} \right]$$

or, since $M < K$,

$$| u(x, y) | \leqq \frac{\mu \cdot (3K)^n(x + y)^n}{n!} \leqq \frac{\mu(3K)^n(a + b)^n}{n!},$$

for every n. Increasing n indefinitely, it follows that $u(x, y) = 0$ or $\varphi_1(x, y) = \varphi(x, y)$ in the rectangle (a, b).

3. As an application consider the partial differential equation

$$\frac{\partial^{m+n}z}{\partial x^m \partial y^n} = \sum_{\nu=0}^{n-1} A_{m\nu}(x, y)\frac{\partial^{m+\nu}z}{\partial x^m \partial y^\nu} + \sum_{\mu=0}^{m-1} A_{\mu n}(x, y)\frac{\partial^{\mu+n}z}{\partial x^\mu \partial y^n} + \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} A_{\mu\nu}(x, y)\frac{\partial^{\mu+\nu}z}{\partial x^\mu \partial y^\nu} + A(x, y), \tag{10}$$

where all the coefficients are bounded and have their discontinuities regularly distributed in the rectangle (a, b). We shall prove that this equation has one and only one solution $z(x, y)$, which, together with all its derivatives occurring in (10), is bounded in the rectangle (a, b) and satisfies the boundary conditions

$$\begin{aligned} &\frac{\partial^\mu z}{\partial x^\mu} = Y_\mu(y), \qquad (\mu = 0, 1, \cdots, m-1) \quad \text{for } x = 0, \quad 0 \leqq y \leqq b, \\ &\frac{\partial^\nu z}{\partial y^\nu} = X_\nu(x), \qquad (\nu = 0, 1, \cdots, n-1) \quad \text{for } y = 0, \quad 0 \leqq x \leqq a, \end{aligned} \tag{11}$$

where the $Y : s$ and their derivatives up to the order n, as well as the $X : s$ and their derivatives up to the order m, are continuous and satisfy the conditions*

$$c_{\mu\nu} = \frac{d^\mu X_\nu(x)}{dx^\mu} = \frac{d^\nu Y_\mu(y)}{dy^\nu} \quad \begin{pmatrix} \mu = 0, 1, \cdots, m-1 \\ \nu = 0, 1, \cdots, n-1 \end{pmatrix} \text{for } x = y = 0. \tag{12}$$

Let $z(x, y)$ be any function satisfying the boundary conditions (11) and (12), and write

$$\frac{\partial^{m+n}z(x, y)}{\partial x^m \partial y^n} = \varphi(x, y). \tag{13}$$

Then the following formulas are readily proved by complete induction:

$$\begin{aligned} \frac{\partial^{m+\nu}z}{\partial x^m \partial y^\nu} &= \sum_{\beta=0}^{n-1-\nu} \frac{d^m X_{\nu+\beta}(x)}{dx^m}\frac{y^\beta}{\beta!} + \int_0^y \varphi(x, t)\frac{(y-t)^{n-1-\nu}}{(n-1-\nu)!}dt, \\ \frac{\partial^{\mu+n}z}{\partial x^\mu \partial y^n} &= \sum_{\alpha=0}^{m-1-\mu} \frac{d^n Y_{\mu+\alpha}(y)}{dy^n}\frac{x^\alpha}{\alpha!} + \int_0^x \varphi(s, y)\frac{(x-s)^{m-1-\mu}}{(m-1-\mu)!}ds, \\ \frac{\partial^{\mu+\nu}z}{\partial x^\mu \partial y^\nu} &= \sum_{\alpha=0}^{m-1-\mu} \frac{d^\nu Y_{\mu+\alpha}(y)}{dy^\nu}\frac{x^\alpha}{\alpha!} + \sum_{\beta=0}^{n-1-\nu} \frac{d^\mu X_{\nu+\beta}(x)}{dx^\mu}\frac{y^\beta}{\beta!} - \sum_{\alpha=0}^{m-1-\mu}\sum_{\beta=0}^{n-1-\nu} c_{\alpha\beta}\frac{x^\alpha y^\beta}{\alpha!\beta!} \\ &\quad + \int_0^x ds \int_0^y \varphi(s, t)\frac{(x-s)^{m-1-\mu}(y-t)^{n-1-\nu}}{(m-1-\mu)!(n-1-\nu)!}dt. \end{aligned} \tag{14}$$

Substituting these values of the derivatives in the equation (10), it is seen immediately that we obtain an integral equation of the form (1), which has a unique bounded solution $\varphi(x, y)$ and consequently (14) gives, making $\mu = \nu = 0$, a unique solution of (10) with the required properties.

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* For the ordinary treatment of this problem, see for instance Note I (by Picard) in vol. 4 of Darboux's Theorie des Surfaces.

THE LINEAR CONTINUUM IN TERMS OF POINT AND LIMIT.*

By Robert L. Moore.

Introduction.

F. Riesz† has proposed a set of four postulates for a continuum in terms of point and limit in the non-sequential‡ sense. To these he adds certain postulates involving the third undefined notion "*verkettung.*" His postulates hold true for continua of any number of dimensions and do not form a categorical set. In the present paper I propose a categorical set of eight postulates for the linear continuum§ in terms of *point* and *limit.* The first three of these postulates (or axioms) are substantially equivalent to the first three of Riesz' system.

The treatment of the linear continuum contained in the present paper connects with Lennes' characterization of a continuous arc AB as a closed, connected set of points containing A and B but containing no connected proper subset that contains A and B.

§ 1. Axioms and Definitions.

Axiom 1. *If the point set M contains the point set N then every limit point of N is also a limit point of M.*

Axiom 2. *If M and N are two point sets with no point in common then every limit point of $M + N$‖ is either a limit point of M or a limit point of N.*

* A part of this paper, under a different title, was presented to The American Mathematical Society, April 26, 1914.

† F. Riesz, Stetigkeitsbegriff und Mengenlehre, Atti del IV Congresso Internazionale dei Matematici, Roma, 1908, vol. 2, pp. 18–24.

‡ In his thesis, Frechet postulates limit in the sequential sense. Cf. M. Frechet, Sur quelques points du calcul fonctionnel, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 6. See also E. R. Hedrick, On properties of a domain for which any derived set is closed, Transactions of the American Mathematical Society, vol. 12 (1911), pp. 285–294; and T. H. Hildebrandt, A contribution to the foundations of Frechet's Calculfonctionnel, American Journal of Mathematics, vol. 34 (1912), p. 237.

§ Postulates for the linear continuum in terms of point and order have been given by O. Veblen, Definition in terms of order alone in the linear continuum and in well ordered sets, Transactions of the American Mathematical Society, vol. 6 (1905), pp. 165–171, and by E. V. Huntington, A set of postulates for a one-dimensional continuum and for the theory of groups, pages 17–41 of the same volume. See also R. E. Root, Limits in terms of point and order, Transactions, vol. 15 (1914), pp. 51–71.

‖ In the language of point set theory, if M and N are two point sets then $M + N$ denotes the set of all points that belong to either M or N. If M contains N and at least one additional point then $M - N$ denotes the set of all points that belong to M and not to N.

AXIOM 3. *No point is a limit point of a single point.*

DEFINITION 1. The point set K is said to be *connected* if, however it be divided into two mutually exclusive* sets, one of those sets contains a limit point of the other one.

AXIOM 4. *If P is a point and S is the set of all points, then $S - P$ is the sum of two mutually exclusive connected sets neither of which contains a limit point of the other.*

DEFINITION 2. If P is a point and S is the set of all points, and $S - P = S_P' + S_P''$, where S_P' and S_P'' are mutually exclusive connected point sets such that neither of them contains a limit point of the other one, then S_P' and S_P'' are called rays. If B is a point of the ray S_P', then the ray S_P' is denoted by PB. The ray S_P' is said to be complementary to the ray S_P''.

It follows from Axiom 1 that if $P \neq B$ then there is not more than one ray PB.

AXIOM 5. *There do not exist three mutually exclusive rays.*

AXIOM 6. *The set of all points is a connected set.*

AXIOM 7.† *There exists a countable set of points such that every point either belongs to this set or is a limit point of it.*

AXIOM 8. *There exist at least two points.*

In the following AB denotes the ray AB. S denotes the set of all points.

§ 2. Consequences of Axioms 1, 2, 3, 4 and 6.

THEOREM 1. *If $B \neq C$, then B is a limit point of the ray BC.*

Proof. By Axiom 4 there exist two points, C and C', such that $S - B = BC + BC'$. Since $S = (B + BC') + BC$, therefore, by Axiom 6, either $S + BC'$ contains a point P which is a limit point of BC or BC contains a point P' which is a limit point of $B + BC'$. But in the latter case, by Axiom 2, either P' is a limit point of B, which contradicts Axiom 3, or P' is a limit point of BC' which is contrary to hypothesis and Definition 2. Hence $B + BC'$ contains a point P which is a limit point of BC. But BC' can contain no limit point of BC. Hence, by Axiom 2, B is a limit point of BC.

THEOREM 2. *If $A \neq B$ and A is not on BC, then AB contains BC.*

Proof. If AB does not contain BC then since A is not a point of BC, BC must contain at least one point in common with AB', the complement

* Two or more point sets are said to be mutually exclusive if no two of them have a point in common.

† Compare Axiom 7 with Frechet's definition of a classe séparable, loc. cit., p. 23. Cf. also G. Cantor, Zur Begründung der transfiniten Mengenlehre, I Mathematische Annalen, vol. 46 (1895), p. 510. Veblen's Postulate of Uniformity is *in some respects* weaker than Axiom 7. Cf. O. Veblen, loc. cit., p. 166.

of AB. Let K be the set of all such common points. There are two cases to be considered.

Case I. Suppose that every point of BC is a point of AB'. Then, by Theorem 1, and Axiom 1, B is a limit point of AB' which is contrary to Definition 2.

Case II. Suppose BC contains at least one point that does not belong to AB'. Let K' denote the set of all such points. Then since BC is connected, either K contains a limit point of K' or K' contains a limit point of K. But K is a subset of AB' and, since by hypothesis A is not on BC and therefore is not in K' and furthermore no point of AB' is in K', therefore K' is a subset of AB. Thus again in case II one of the rays AB and AB' would contain a limit point of the other one. But this is contrary to Definition 2.

THEOREM 3. *If $B \neq A$, then every point P lies either on BA or on AB.*

Proof. If P is not on AB, then either P is A or AB and AP are two complementary rays. Hence either P is A or B is not on AP. Therefore, by Theorem 2, BA contains P.

DEFINITION 3. Three points A, B, C are said to be in the order ABC if and only if they are mutually distinct and A is not on BC. The abbreviation "ABC," used as a sentence, signifies that A, B and C are in the order ABC.

THEOREM 4. *If ABC, then CBA.*

Proof. If C were on BA, then BC would be identical with BA and A would be on BC, which is contrary to hypothesis.

THEOREM 5. *If ABC, then not BCA.*

Proof. If B were not on CA, then, by Theorem 2, BC would contain A, which is contrary to hypothesis.

THEOREM 6. *If ABC and BCD, then ACD.*

Proof. Since A is not on BC and B is not on CD, therefore, by Theorem 2, AB contains BC and BC contains CD. Hence AB contains CD. Therefore, if A were on CD, AB would contain A, which is contrary to Definition 2.

THEOREM 7. *If ABC and ACD, then ABD.*

Proof. By hypothesis and Theorem 2, AB contains BC and AC contains CD. Hence AB is identical with AC and AC contains D. Therefore AB contains D. Suppose now that ABD were not true. Then, by Theorem 4, Theorem 5 and Definition 3, D would be on BA. But, by hypothesis, ABC. Hence, by Theorems 2 and 4, CB contains BA and therefore CA contains BA. Hence if D were on BA, D would be on CA. Hence A would be on CD which is contrary to the hypothesis that ACD. Hence D is not on BA. Hence DBA and therefore ABD.

THEOREM 8. *If ABC and BCD, then ABD.*

Proof. From *ABC* and *BCD* it follows, by Theorem 6, that *ACD*. From *ACD* and *ABC* it follows, by Theorem 7, that *ABD*.

THEOREM 9. *If ABC and ACD, then BCD.*

Proof. By hypothesis and Theorem 7, *ABD*. If *BCD* were not true, then *B* would lie on *CD*. Hence *CB* would be identical with *CD*. But *ACD*. Therefore *A* is not on *CD*. Hence *A* would not be on *CB*. But *ABC*. Therefore *CBA*. Hence, by Theorem 2, *A* is on *CB*. Thus the supposition that *B* is on *CD* leads to a contradiction.

DEFINITION 4. The abbreviation *ABCD* signifies that *ABC*, *ABD*, *ACD* and *BCD*.

THEOREM 10. *If ABC and ACD, or if ABC and BCD, then ABCD.*

THEOREM 11. *If A and B are distinct points, there exists a point C such that ABC.*

Proof. By Axiom 4 there exists at least one point *C* different from *B* and not on the ray *BA*. Hence *CBA* and therefore *ABC*, by Theorem 4.

§ 3. Consequences of Axioms 1–6.

THEOREM 12. *If ABC and ABD, and C ≠ D, either BCD or BDC.*

Proof. By hypothesis, Theorem 4 and Definition 3, *BA* contains neither *C* nor *D*. By Theorem 2, *BA* is in *CB* and in *DB*. Suppose now that neither *BCD* nor *BDC*. Then *B* is on *CD* and on *DC* and therefore *CB* is *CD* and *DB* is *DC*. Hence *BA* is on *CD* and on *DC*. By Theorem 11 there exist points *C'* and *D'* such that *CDC'* and *DCD'*. By Theorem 2, *DC* contains *CD'*. Hence if *CD'* and *DC'* had a point in common then *DC* and *DC'* would have a point in common, which contradicts *CDC'*. But *BA* lies in *DC* and in *CD*. Hence it can have no point in common with either *DC'* or *CD'*. Thus no two of the three rays *BA*, *CD'* and *DC'* have a point in common. But this is contrary to Axiom 5. Hence the supposition that neither *BCD* nor *BDC* leads to a contradiction.

THEOREM 13. *If A and C are two distinct points, there exists a point B such that ABC.*

Proof. By Theorem 11 there exists a point *D* such that *ACD*. By Theorem 4, *DCA*. By Theorem 1, *A* is a limit point of *AC*. But since *A* is not *C* and *A* is not on *CD*, therefore *A* is not a limit point of *CD*. Hence, by Axioms 1, 2 and 3, *AC* must contain at least one point *B* which is different from *C* and does not belong to *CD*. Hence *BCD* and therefore *DCB*. From *DCB* and *DCA* it follows by Theorem 12 that either *CAB* or *CBA*. But *B* is on *AC*. Hence it is not true that *BAC*. Therefore *CAB* is false. Hence *CBA* and therefore *ABC*.

THEOREM 14. *If A, B and C are three distinct points, then either ABC, BCA or CAB.*

Proof. Suppose that ABC is not true. Then CBA is not true and therefore C is on BA. By Theorems 11 and 4 there exists a point A' such that $A'BA$. Hence A' is not on BA. But since C is on BA, therefore BA is BC. Therefore A' is not on BC. Hence $A'BC$. Hence, by Theorem 12, either BCA or BAC. Thus it is proved that if ABC is not true, then either BCA or BAC is true.

It is now easy to see that if Axioms 1–6 hold true for a space S then all of Hilbert's linear Axioms of Group II* hold true in S with respect to the betweenness defined in Definition 2. In particular it may be shown that any four distinct points may be so lettered A, B, C, D that $ABCD$. If Axioms 7 and 8 are added, my set of axioms becomes† categorical with respect to point and betweenness *as defined* in Definition 2. From this fact it is not‡ *apriori* evident that the set of Axioms 1–8 is *absolutely* categorical (i. e. categorical with respect to point and *limit*, the undefined symbols in terms of which these Axioms are stated). It can however be proved that a point P is a limit point of a point set M if, and only if, every segment that contains P contains at least one point of M different from P. Thus the statement that P is a limit point of M is (in the presence of Axioms 1–8 and Definition 2) equivalent to a statement in terms of point and betweenness. It follows that our set of Axioms is categorical with respect to *point* and *limit* point of a point set.

§ 4. Concerning Axiom 5.

Lemma 1. *It follows from Axioms 1, 2, 3, 4, 6 and Theorems 12 and 13 that if* $A \neq C$ *and the two rays* AB *and* CD *have no point in common, then* $BACD$.

Proof. By hypothesis, Theorem I, and Axioms 1 and 3 B is not on CD and D is not on AB. Hence BCD and DAB. By Theorem 14§ either DCA, CAD or ADC.

Suppose that ADC were true. Then CDA. But DAB. Hence CDB, by Theorem 8. Thus ADC is impossible.

Suppose that CAD. By Theorem 13 there exists a point X such that CXA. From CXA and CAD it follows, by Theorem 7, that CXD. From CAD and BCD it follows by Theorems 4 and 9 that ACB. But AXC. Hence, by Theorem 7, AXB. But from CXD and AXB it follows that AB and CD have the point X in common. This is contrary to hypothesis. Thus CAD is false.

* D. Hilbert, The Foundations of Geometry, Translation by E. J. Townsend, Chicago, 1902, pp. 5–7.

† Cf. G. Cantor, loc. cit.

‡ Cf. § 7.

§ It was shown above that Theorem 14 is a consequence of Axioms 1, 2, 3, 4, 6 and Theorem 12.

It follows that DCA is true. Similarly BAC is true. Hence $BACD$.

LEMMA 2. *Axiom 5 is a Consequence of Axioms 1, 2, 3, 4, 6 and Theorems 12 and 13.*

Proof. Suppose that AB, CD and EF are three mutually exclusive rays. It is clear that A, C and E must be mutually distinct. Hence, by Lemma 1, $BACD$, $BAEF$, $DCEF$ and $DCAB$. From $DCEF$ it follows that $F \neq C$. By Theorem 12, from $BACD$ and $BAEF$ it follows that ACF or AFC and from $DCEF$ and $DCAB$ it follows that CFA or CAF. But, by Theorems 4 and 5, ACF and CAF cannot both be true. Hence, by Theorem 4, AFC. From CAB and AFC it follows, by Theorems 4 and 7, that CFB. From CFB and CEF it follows that EFB and therefore that EF contains B which is contrary to hypothesis. Thus Axiom 5 is proved on the basis of Axioms 1, 2, 3, 4, 6 and Theorems 12 and 13.

THEOREM 15. *In the presence of Axioms 1, 2, 3, 4 and 6, Axiom 5 is equivalent to Theorems 12 and 13 combined.*

§ 5. Questions of Independence.

According to E. H. Moore a set of postulates T is said to be *completely* independent if

(1) For each subset of T there exists a system in which all the postulates of that subset are satisfied and the remaining postulates of T are not satisfied.

(2) There exists a system in which no postulate of the set T is satisfied.

Since the denial of Axiom 8 implies that S contains not more than one point, it would seem very improbable *apriori* that the set 1–8 should be completely independent. If 8 be omitted however the remaining set, G, of Axioms 1–7 is completely independent except that there exist no examples of type E_{567} or E_{2567}. Here the notation E_{567} is used to denote an example of a system in which 5, 6, and 7 are false and the remaining axioms 1, 2, 3 and 4 are true. In general, the letter E with subscripts denotes an example of a system in which those axioms are false whose numbers occur as subscripts of E and the remaining axioms of the set G are true.

The following notation will be used.

K_1 denotes a segment without end points.

K_2 denotes a segment with end points.

K_3 denotes the set of all the interior points of two circles which have no interior point in common.

K_4 denotes the set of all points, except O, on and interior to a circle R with center at O.

K_5 denotes the set of all interior points of two colinear segments which have no point in common.

K_6 denotes a circle C together with three rays DD', EE' and FF' where D, E and F are distinct points of C, and D', E' and F' are three points such that ODD', OEE' and OFF' where O is the center of C.

K_7 is the same as K_6 except that in K_7 all points X such that $OD'X$ are omitted.

K_8 denotes a segment t together with two distinct points neither of which is a point or an end point of t.

K_9 denotes a segment t together with three distinct points no one of which is a point or an end point of t.

For the sake of convenience the following independence examples will be separated into groups.

In each example of group I there are in S three distinct points, A, B and C such that BAC in the ordinary sense and such that: (1) A is a *limit* point of a point set M if, and only if, M contains B and C, or M contains A, or A is a limit point of M in the usual sense.

(2) If $P \neq A$, then P is a *limit* point of M only if it is a limit point of M in the usual sense.

GROUP I.

E_{23}. S is K_1, E_{2345}. S is K_6, E_{2346}. S is K_5,
E_{234}. S is K_2, E_{236}. S is K_3, E_{23456}. S is K_8.

In each example of group II, a point P is a *limit* point of a point set M if, and only if,

(1) P belongs to M or is a limit point of M in the ordinary sense.

(2) M contains at least two points.

GROUP II.

E_2. S is K_1, E_{245}. S is K_6, E_{2456}. S is K_8,
E_{24}. S is K_2, E_{246}. S is K_5, E_{26}. S is K_3.

In each example of group III, P is a limit point of M if, and only if, P is a limit point of M in the usual sense or P belongs to M.

GROUP III.

E_3. S is K_1, E_{36}. S is K_3, E_{346}. S is K_9,
E_{34}. S is K_2, E_{345}. S is K_6, E_{3456}. S is K_8.

In each example of group IV, a point P is a *limit* point of a point set M if, and only if, P is a limit point of M in the usual sense.

GROUP IV.

E_4. S is K_2, E_{45}. S is K_6, E_{456}. S is K_8.
E_6. S is K_3, E_{46}. S is K_5,

Group V.

E_{235}. S is K_4. There are three points A, B and C within R such that $OBAC$ in the ordinary sense and such that

(1) A is a *limit* point of a point set M if, and only if, M contains B and C, or M contains A, or A is a limit point in the usual sense of a subset of M which lies on the ray OA.

(2) If $P \neq A$ then P is a *limit* point of M if, and only if, P is a limit point, in the usual sense, of a subset of M that lies on the ray OP, or of a subset of M that lies on R.

E_{2356}. S is composed of the interior, K, of a circle together with two distinct points A and B that lie without that circle. A is a *limit* point of M if, and only if, M contains B. If P is a point in K, then P is a *limit* point of the point set M if, and only if, M contains two or more points of K. B is a *limit* point of no point set.

E_{25}. S is K_4. P is a *limit* point of M if, and only if, (1) P belongs to M or is a limit point in the usual sense of a subset of M that lies entirely on R or entirely on OP, (2) M contains at least two points.

E_{35}. S is K_4. P is a *limit* point of M if, and only if, P belongs to M or is a limit point in the usual sense of a subset of M that lies entirely on R or entirely on the radius OP.

E_5. S is K_4. P is a limit point of M if, and only if, P is a limit point in the usual sense of a subset of M that lies entirely on R or entirely on the radius OP.

E_{356}. S is the same as in E_{2356}. P is a *limit* point of M if, and only if, either, (1) P belongs to K and is a limit point of M in the usual sense or, (2) $P = A$ and M contains B.

Group VI.

E_{56}. S is composed of three points no one of which is a *limit* point of any point or set of points.

E_{256}. S is composed of three points A, B and C. P is a *limit* point of M if, and only if, $P = A$ and M is $A + B$ or $A + B + C$.

Let E denote any one of the preceding examples. Let E′ denote an example which is the same as E except that in E′ no point is a *limit* point of S. Then every axiom of the set G which is satisfied in E is also satisfied in E′ and conversely, except that Axiom 1 is not satisfied in E′.

Let $\overline{\text{E}}$ denote any one of the preceding examples (examples of type E′ being included) with the exception of the examples E_{56}, E_{256}, E_{56}' and E_{256}'. In $\overline{\text{E}}$ either S is an ordinary continuum of one or two dimensions, or it contains such a continuum. If a linear continuum occurs in $\overline{\text{E}}$ replace it by a system of the sort described in Veblen's independence example for

his uniformity Postulate E.* If a two dimensional continuum exists in $\overline{\mathrm{E}}$ replace it by a certain two-dimensional set† which may be easily constructed with the help of the above-mentioned linear set of Veblen's. Let $\overline{\mathrm{E}}'$ denote an example which is the same as $\overline{\mathrm{E}}$ except for the above mentioned substitution and certain modifications which naturally accompany that substitution. Then every postulate of the set G which holds true in $\overline{\mathrm{E}}$ holds true also in $\overline{\mathrm{E}}'$ and conversely, except that 7 is not true in $\overline{\mathrm{E}}'$.

We have yet to consider the existence or non-existence of E_{1567}, E_{12567}, E_{567} and E_{2567}. The first two of these exist.

E_{1567}. S is composed of three mutually exclusive subsets S_1, S_2, S_3. A *point* is a system of four real numbers $(x_1, y_1; x_2, y_2)$ such that $0 \leqq y_k \leqq 1$ and such that either $0 < x_k < 1$, $2 < x_k < 3$ or $4 < x_k < 5$ $(k = 1, 2)$. In the first case $(x_1, y_1; x_2, y_2)$ belongs to S_1, in the second case to S_2 and in the third case to S_3. The point $(x_1', y_1'; x_2', y_2')$ is a *limit* point of the point set M if, and only if, one of the following two conditions is fulfilled.

Condition I. Corresponding to any preassigned positive number ϵ there exists, in the set M, a point $(x_1'', y_1''; x_2'', y_2'')$, distinct from $(x_1', y_1'; x_2', y_2')$, such that (1) $|x_k' - x_k''| < \epsilon$, $|y_k' - y_k''| < \epsilon$ $(k = 1, 2)$ (2) $|x_k' - x_k''| = 0$ in case y_k' is distinct from 0 and from 1 $(k = 1, 2)$.

Condition II. The point $(x_1', y_1'; x_2', y_2')$ belongs to the set S_i, and M is a non-denumerable proper subset of S_{i+1} (mod. 3) $(i = 1, 2, 3)$.

E_{12567}. This example is the same as E_{1567} except that in E_{12567} every point P in S_1 is a limit point of every pair of points in S_1 which are distinct from each other and from P.

THEOREM 16. *If Axioms 5 and 6 are false and Axioms 1, 3 and 4 are true, then S must consist of just three points.*

Proof. Since Axiom 6 is false, S must consist of two subsets S_1 and S_2 neither of which contains a limit point of the other. Since Axiom 5 is false, there exist three mutually exclusive rays r_1, r_2, r_3. Hence one of the sets S_1 and S_2 contains at least two points. If S_1 contains more than one point and P is a point of S_1 then it easily follows from Axioms 4 and 1 that P divides S into the two rays $S_1 - P$ and S_2 and therefore S_2 is connected. Similarly if S_2 contains more than one point then S_1 is connected. Let P_1, P_2, P_3 denote the origins of the rays r_1, r_2, r_3 respectively. At least two of these points (say P_1 and P_2) must be distinct from each other. There are two cases to consider.

Case I. Suppose P_1 and P_2 are in S_1. Then, by Axioms 4 and 1,

$$r_1 \text{ is either } S_1 - P_1 \text{ or } S_2, \qquad r_2 \text{ is either } S_1 - P_2 \text{ or } S_2.$$

* O. Veblen, loc. cit., p. 169.

† The set S_1, described below in connection with E_{1567}, may be used for this purpose.

But r_1, r_2 and r_3 are mutually exclusive. It clearly follows that $S_1 = P_1 + P_2$. But, by Axiom 3, $P_1 + P_2$ is not connected. Hence S_2 must consist of only one point. Therefore $S = P_1 + P_2 + P_3$.

Case II. Suppose P_1 and P_3 are in S_1 and P_2 is in S_2. Then

$$r_1 \text{ is } S_1 - P_1 \text{ or } S_2, \qquad r_2 \text{ is } S_2 - P_2 \text{ or } S_1, \qquad r_3 \text{ is } S_1 - P_3 \text{ or } S_2.$$

But r_1, r_2 and r_3 are mutually exclusive. Hence r_1 is $S_1 - P_1$, r_2 is $S_2 - P_2$ and r_3 is $S_1 - P_3$. Hence $S_1 - P_1$ and $S_1 - P_3$ have no point in common and therefore S_1 is $P_1 + P_3$ and is therefore not connected. Hence S_2 contains only one point, P_2. Therefore $S = P_1 + P_3 + P_2$. Our theorem is thus established.

It follows from the above theorem that if Axioms 5 and 6 are false and Axioms 1, 3 and 4 are true, then Axiom 7 is necessarily true. Hence there exist no examples of type E_{567} or E_{2567}.

That the set of Axioms 1–8 is independent in the sense that no one of these axioms follows from the others is shown by the examples E_1, E_2, E_3, E_4, E_5, E_6 and E_7 together with the following example, E_8.

E_8. S is a single point. This point is a limit point of no point-set.

§ 6. Concerning Axiom 4.

For Axiom 4 may be substituted the following two Axioms, 4(a) and 4(b).

Axiom 4(a). *If P is a point and S is the set of all points, then $S - P$ is not connected.*

Axiom 4(b). *If P is a point and S is the set of all points, then $S - P$ is composed of two connected subsets.*

Theorem 17. *Axiom 4 is a consequence of Axioms 1, 4(a) and 4(b).*

Proof. By Axiom 4(b), $S - P$ is composed of two connected subsets, S_1 and S_2. By Axiom 4(a), $S - P$ is composed of two subsets M_1 and M_2 neither of which contains a limit point of the other. Suppose S_1 contains a point of M_1 and also a point of M_2. Let S_1' denote the set of all points of M_1 which lie in S_1 and let S_2' denote the set of all points of M_2 which lie in S_1. Then, since S_1 is connected, one of the sets S_1' and S_2' must contain a limit point of the other one and therefore, by Axiom 1, one of the sets M_1 and M_2 must contain a limit point of the other one. But this is contrary to hypothesis. It follows that one of the sets S_1 and S_2 coincides with M_1 and the other one coincides with M_2. Hence neither of the sets S_1 and S_2 contains a limit point of the other one. Thus Axiom 4 is established.

Let G' denote the set of Axioms 1–3, 5–8, 4(a) and 4(b).

To see that 4(b) is not a consequence of the remaining axioms of the

set G', consider the example in which S denotes two intersecting straight lines in an ordinary euclidean plane and in which the terms point and limit have their usual significance. In this example, $4(b)$ is false but all the other axioms of the set G' are true.

The independence of $(4a)$ in the set G' is exhibited by E_4.

It follows that G' is a categorical set of independent axioms.

§ 7. Concerning Categoricity.

In § 3 it was indicated that a set of axioms may be categorical with respect to certain defined terms but, at the same time, not *absolutely** categorical. I will mention two interesting examples. Let H_{pl} denote the set of axioms composed of Hilbert's plane axioms of groups I and II together with a Dedikind cut continuity postulate and an axiom P to the effect that all points are coplanar. The set H_{pl} is not categorical with respect to the undefined symbols, point, line, association and betweenness, in terms of which these axioms are stated. Let V_{pl} denote the set of axioms composed of Veblen's Axioms I–VIII, XI together with axiom P (cf. above). The set V_{pl} is not categorical (with respect to point and order). But both H_{pl} and V_{pl} are categorical with respect to *point* and *limit* point of a point-set as defined in Veblen's Theory of Plane Curves in Non-Metrical Analysis Situs.†

The truth of the above statements concerning the sets H_{pl} and V_{pl} may be easily seen with the aid of results established in my paper, On a Set of Postulates Which Suffice to Define a Number-plane.‡

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* Cf. § 3.

† Transactions of the American Mathematical Society, Vol. 6 (1905), p. 85.

‡ Ibid., Vol. XVI (1915), pp. 27–32.

A PLANE CUBIC CREMONA TRANSFORMATION AND ITS INVERSE.*

By Dr. F. M. Morgan.

1. In the present paper attention is called to a very simple cubic Cremona transformation of the plane in which a fundamental point A_1 coincides with a fundamental point A_2, where by A_i is understood a fundamental point of multiplicity i. Such an example of a Cremona transformation is of interest because in 1901 Segre† discovered that Noether's‡ proof that every Cremona transformation can be expressed in terms of quadratic transformations is not valid when two fundamental points A_i and A_k come into coincidence. A short time later Castelnuovo§ completed the proof so as to apply to this case as well.

It is also shown in this paper how this cubic Cremona transformation with coincident fundamental points gives a simple ruler and compass construction for some well-known higher plane curves.

Zahradink,|| Aubert and Papelier¶ give the cubic transformation

$$x_1 = \frac{xy^2}{x^2 + y^2}, \qquad y_1 = \frac{x^2 y}{x^2 + y^2}, \tag{1}$$

and its inverse

$$x = \frac{x_1^2 + y_1^2}{x_1}, \qquad y = \frac{x_1^2 + y_1^2}{y_1}. \tag{2}$$

Zahradink uses these transformations in considering the cubic curves $(x_1^2 + y_1^2)(ay_1 + bx_1) + cx_1y_1 = 0$ and $axy^2 + byx^2 + c(x^2 + y^2) = 0$, while Aubert and Papelier use them in the following exercise.

Exercise. Consider two rectangular axes OX and OY and any point $M(x, y)$ in the plane. (See figure.) Draw MP and MQ perpendicular to OX and OY respectively. Draw PR and QS perpendicular to OM. Through R and S draw parallels to OX and OY respectively, meeting in U. If the coördinates of U are x_1 and y_1, derive the transformations (1) and (2). By means of these transformations find the transforms of a straight line and of a circle passing through the origin.

* Read before the American Mathematical Society, September 8, 1914.

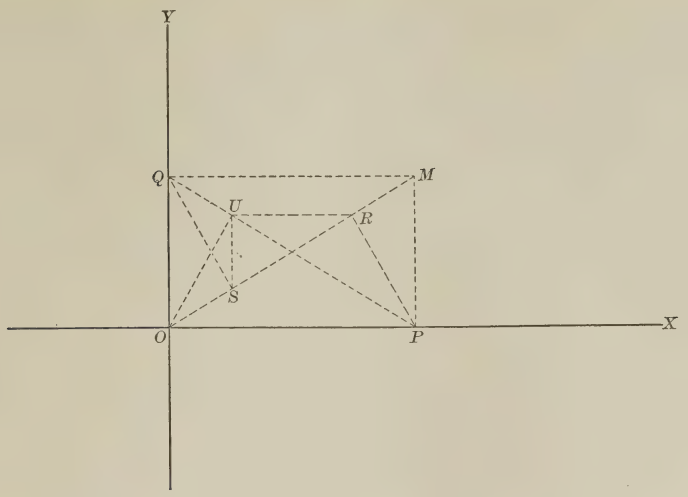
† Atti della Reale Accademia di Torino, vol. 36 (1901), pp. 645–651.

‡ Mathematische Annalen, vol. 3 (1871), p. 164.

§ Atti della Reale Accademia di Torino, vol. 36 (1901), pp. 861–874.

|| Casopis, vol. 34 (1905), pp. 329–340.

¶ Exercises de Géométrie Analytique, vol. 1, pp. 152–153.



The properties of these transformations do not seem to have been discussed further. These may be found much more easily if the transformations are made homogeneous. Calling them S and T, we have

$$S = \left\{ \begin{matrix} x & y & z \\ y_1(x_1^2 + y_1^2)\,, & x_1(x_1^2 + y_1^2)\,, & x_1y_1z_1 \end{matrix} \right\},$$

$$T = \left\{ \begin{matrix} x_1 & y_1 & z_1 \\ xy^2\,, & yx^2\,, & (x^2 + y^2)z \end{matrix} \right\}.$$

Now if we let

$$I = \left\{ \begin{matrix} x_1 & y_1 & z_1 \\ x_2z_2\,, & y_2z_2\,, & x_2^2 + y_2^2 \end{matrix} \right\},\quad R = \left\{ \begin{matrix} x_2 & y_2 & z_2 \\ yz & xz & xy \end{matrix} \right\}$$

it follows that $T = RI$. Since however $S^{-1} = T$, we have $S^{-1} = RI$ or $S = I^{-1}R^{-1}$. But R, I, R^{-1}, I^{-1}, are all quadratic transformations, and therefore we have found the quadratic transformations in terms of which S and T may be expressed.

2. The Fundamental System of the Plane S. A cubic Cremona transformation* has $\alpha_2 = 1$, $\alpha_1 = 4$ where α_i means the number of i-fold fundamental points. In any Cremona transformation it is known that the Jacobian curve for the fundamental system is identical with the totality of fundamental curves of the net.* The Jacobian in the plane S is $J = x_1y_1(x_1^2 + y_1^2) = 0$. Thus the fundamental points are $A_2 = (0, 0, 1)$, $A_1^{(1)} = (1, i, 0)$, $A_1^{(2)} = (1, -i, 0)$ and $A_1^{(3)} = (0, 0, 1)$ counted twice. The fundamental curves are four straight lines connecting A_2 to each of the four points A_1, and the conic $x_1^2 + y_1^2 = 0$ which passes through all

* Doehlemann: Geometrische Transformationen, Part 2, pp. 142–149.

five fundamental points, (0, 0 1). The point is thus seen to be the union of A_2 and $2A_1$.

3. The Fundamental System of the Plane T. The Jacobian in the plane T is $J = -3x^2y^2(x^2 + y^2) = 0$. The fundamental points are, therefore, $A_2 = (0, 0, 1)$, $A_1^{(1)} = (0, 1, 0)$, $A_1^{(2)} = (1, 0, 0)$ and $A_1^{(3)} = (0, 0, 1)$ counted twice. The fundamental curves are four straight lines connecting A_2 to each of the four points A_1, and the conic $xy = 0$ which passes through all five fundamental points. The point (0, 0, 1) is the union of A_2 and $2A_1$.

4. Geometrical Considerations. *Construction.* A very simple construction in rectangular coördinates for finding the point U, given the point M, is as follows. (See figure.) From M drop perpendiculars MP and MQ to the X and Y axes respectively. Draw the line PQ and drop a perpendicular from O to PQ. The intersection of these two lines is U. This construction has an advantage over that of Aubert and Papelier, in the fact that from this construction it is easy to get M when U is given. For, if we draw UO, erect a perpendicular to it at U and call P and Q the points where it meets the X and Y axes, then OP and OQ are the coördinates of the desired point M.

If the single infinity of curves $x^2 + y^2 = a^2z^2$ is transformed by S, it becomes $(x_1^2 + y_1^2)^3 = a^2x_1^2y_1^2z_1^2$. If this system is first written in non-homogeneous coördinates, and then transformed to polar coördinates it takes the form $2\rho = a \sin 2\theta$ which is the well-known polar equation of a single infinity of four-leaved roses. Therefore using the construction already described, we have a method of constructing a four-leaf rose by means of a ruler and compass.

If on the other hand the system of conics $x_1^2 + y_1^2 = a^2z_1^2$ is transformed by T, the resulting system is $x^2y^2 = a^2(x^2 + y^2)z^2$. In non-homogeneous coördinates the curves of this system are quartic curves with four asymptotes tangent to the circle $x^2 + y^2 = a^2$ and parallel to the axes. The origin is an isolated double point on each curve of the system. The transformation S^2 transforms this system of quartics into the system of roses. T^2 performs the reverse operation.

The system of conics $xy = a^2z^2$ is transformed by S into the system $(x_1^2 + y_1^2)^2 = a^2x_1y_1z_1^2$. This system when reduced to polar coördinates becomes $2\rho^2 = a^2 \sin 2\theta$. Thus from an equilateral hyperbola a lemniscate can be constructed by means of a ruler and compass.

The system of conics $x_1z_1 = a^2y_1^2$ when transformed by S becomes $x(x^2 + y^2) = a^2y^2z^2$ which in non-homogeneous coördinates is at once recognized to be a system of cissoids. Therefore we have a ruler and compass construction for a cissoid.

Thus not only do the transformations (1) and (2) afford examples of Cremona transformations in which two fundamental points coincide, but their geometrical interpretation gives us a ruler and compass construction for some well-known higher plane curves when we start with a conic, a curve constructible by the same process.

DARTMOUTH COLLEGE,
September, 1914.

RELATION BETWEEN THE ROOTS OF A RATIONAL INTEGRAL FUNCTION AND ITS DERIVATIVE.

By Frank Irwin.

In the number of the *Annals* for March, 1914, Hayashi gives a proof, the first, so far as known, not based on dynamical conceptions, of the proposition that if $f(z)$ be a polynomial, and $f'(z)$ its derivative, the roots of $f'(z) = 0$ lie in the complex plane within or on the boundary of the smallest convex polygon that contains all the roots of $f(z) = 0$. It may, however, be noted that the simple proof reproduced, for instance, in Osgood's *Funktionentheorie* (2d ed., p. 210), can readily be modified to avoid the use of the language of dynamics.

We are to consider the values of z that make $f'(z)/f(z)$, that is

$$\frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n},$$

vanish, $z_1, \cdots z_n$ being the roots of $f(z) = 0$; or, what is the same thing, that make

$$\frac{1}{\bar{z} - \bar{z}_1} + \cdots + \frac{1}{\bar{z} - \bar{z}_n}$$

vanish, where $z, \bar{z}_1, \cdots$ are the conjugates of $z, z_1, \cdots$. Now the vector $1/(\bar{z} - \bar{z}_i)$ has the same direction as $z - z_i$. If then z lie outside the polygon referred to in the enunciation of the proposition, say on the other, the outer side of the side AB of that polygon, each of the vectors $1/(\bar{z} - \bar{z}_1), \cdots, 1/(z - z_n)$, will be directed from inside the polygon, across AB, to outside the polygon, in other words their components perpendicular to AB will all have the same sense (and none of them will be zero); and the sum of these vectors cannot, therefore, be zero.

Berkeley, California,
October, 1914.

ERRATUM.

Volume 16.

Some remarks on conformal representation. By T. H. Gronwall. Page 75, line 10, read $\pi\left(\frac{1}{r^2} - \sum_{n=1}^{\infty} n\,|a_n|^2 r^{2n}\right)$ instead of $\pi\left(1 - \sum_{=1}^{\infty} n\,|a_n|^2 r^{2n}\right)$.

PROJECTIVE CLASSIFICATION OF CUBIC SURFACES MODULO 2.

By L. E. Dickson.

1. A preliminary classification is given by the number N of real points (i. e., with integral coördinates) on the surface. This number N is always odd (§ 2), while there are surfaces with 1, 3, $\cdots$, 15 real points. We need examine only the non-equivalent sets of N points, equivalence being with respect to linear homogeneous transformation with integral coefficients modulo 2. In treating the surfaces whose real points are those of a fixed set of N points, the automorphs of that set are the transformations available for the specialization of the parameters in the coefficients of the surface.

For each type* of surface without singular points, all of the real straight lines are given; the number of such lines is 15, 9, 5, 3, 2, 1 or 0 (in contrast with the corresponding numbers 27, 15, 7 or 3 for cubic surfaces in ordinary space).

But the classification includes all cubic surfaces, not a cone and not formed in part of a plane.†

2. The general cubic form in x, y, z, w is

$$S = ax^3 + by^3 + cz^3 + dw^3 + lx^2y + fxy^2 + gx^2z + hxz^2 + ix^2w + jxw^2 + ky^2z + \lambda yz^2 + my^2w + nyw^2 + pz^2w + qzw^2 + rxyz + sxyw + txzw + vyzw.$$

Making use of the abbreviations

$$A = l + f, \quad B = g + h, \quad C = i + j, \quad D = k + \lambda,$$

$$E = m + n, \quad F = p + q,$$

* For certain types the configuration of the real and imaginary lines has been considered by the writer in Proc. Nat. Acad. Sciences, April, 1915.

† We do not list the quaternary cubic forms $C(x, y, z, w)$ with a real linear factor or those equivalent to a ternary form $\phi(X, Y, Z)$. A necessary condition for the equivalence of C and ϕ under the linear transformation in which the coefficients of W in x, y, z, w are $\alpha, \beta, \gamma, \delta$ is

$$0 \equiv \frac{\partial \phi}{\partial W} = \alpha \frac{\partial C}{\partial x} + \beta \frac{\partial C}{\partial y} + \gamma \frac{\partial C}{\partial z} + \delta \frac{\partial C}{\partial w}$$

and hence that the first partial derivatives of C be linearly dependent. Attention will be called to the cases in which they are dependent and a special examination made (note that ϕ has the singular point 4). For the detection of a linear factor, note that xQ has as singular points all the intersections of $x = 0$, $Q = 0$.

we see that the values of S at the fifteen real points* in space are

$$1 = (1000) : a \qquad 2 = (0100) : b \qquad 3 = (0010) : c \qquad 4 = (0001) : d$$
$$5 = (1100) : a + b + A \qquad 6 = (1010) : a + c + B$$
$$7 = (1001) : a + d + C \qquad 8 = (0110) : b + c + D$$
$$9 = (0101) : b + d + E \qquad 10 = (0011) : c + d + F$$
$$11 = (1110) : a + b + c + A + B + D + r$$
$$12 = (1101) : a + b + d + A + C + E + s$$
$$13 = (1011) : a + c + d + B + C + F + t$$
$$14 = (0111) : b + c + d + D + E + F + v$$
$$15 = (1111) : a + b + c + d + A + B + C + D + E + F + r + s + t + v.$$

The sum of these linear functions is congruent to zero modulo 2. If N be the number of real points on $S \equiv 0$, N of these functions are congruent to zero and $15 - N$ are congruent to unity. Adding, we see that $0 \equiv 15 - N \pmod 2$. Hence *the number of real points on any cubic surface is odd.*

3. $N = 15$. Let the surface contain all fifteen real points. Then the linear functions in the table of § 2 are all congruent to zero, whence $a = b = c = d = A = B = C = D = E = F = r = s = t = v = 0 \pmod 2$. Excluding the case in which each coefficient of S is zero, we may set $l = 1$, after permuting the variables. Replacing y by $y + gz + iw$, we have $l = 1$, $g = i = 0$. Then replacing x by $x + kz + mw$, we have also $k = m = 0$. If $p = 0$, we have the binary form $x^2y + xy^2$ (three planes). If $p = 1$, we have

$$(1) \qquad xy(x + y) + zw(z + w),$$

which has no singular point; just 15 of its 27 straight lines are real, while each of its 45 sets of 3 coplanar lines are concurrent.

4. $N = 1$. The single real point on the surface can be transformed into 1. Then, a, A, B, C, r, s, t are zero, while b, c, d, D, E, F, v are unity. We exclude the case $l = g = j = 0$, since S is then free of x. Permuting y, z, w, we may set $l = 1$. Replacing y by $y + gz + jw$, we have $l = 1$, $g = j = 0$. Replacing x by $x + kz + mw$, we have also $k = m = 0$. Replacing z by $z + pw$, we have also $p = 0$, and get

$$(2) \qquad y^3 + z^3 + w^3 + x^2y + xy^2 + yz^2 + yw^2 + zw^2 + yzw.$$

There is no singular point and, of course, no real straight line.

* We shall need a list of the 35 sets of 3 collinear real points:

1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	
2	3	4	8	9	10	14	3	4	6	7	10	13	4	5	7	9	12
5	6	7	11	12	13	15	8	9	11	12	14	15	10	11	13	14	15
4	4	4	4	5	5	5	5	6	6	6	7	7	8	8	9	10	
5	6	8	11	6	7	10	13	7	9	12	8	11	9	12	11	11	
12	13	14	15	8	9	15	14	10	15	14	15	14	10	13	13	12.	

5. $N = 13$. We may take 1 and 2 as the two real points not on the surface. Then a, b, B, C, D, E, t, v are unity and c, d, A, F, r, s zero. Thus

$$S = x^3 + y^3 + l(x^2y + xy^2) + gx^2z + (g+1)xz^2 + ix^2w + (i+1)xw^2 + ky^2z + (k+1)yz^2 + my^2w + (m+1)yw^2 + p(z^2w + zw^2) + xzw + yzw.$$

For each of the following transformations, we give the values of the altered coefficients:

$x' = x + z$: $g' = g + 1$, $k' = k + l$, $p' = p + i + 1$;

$x' = x + w$: $i' = i + 1$, $m' = m + l$, $p' = p + g + 1$;

$y' = y + z$: $k' = k + 1$, $g' = g + l$, $p' = p + m + 1$;

$y' = y + w$: $m' = m + 1$, $i' = i + l$, $p' = p + k + 1$.

First, let $l = 0$. Then we can make $g = i = k = m = 0$. If $p = 0$,

$$S = (x + y)(x^2 + xy + y^2 + z^2 + zw + w^2).$$

If $p = 1$, we replace y by $y + x$ and w by $w + y$ and get

$$xy(x + y) = w(w + z)(y + z). \tag{3}$$

It has no singular point. Of its 27 straight lines, the only real ones are the nine obtained by equating to zero a factor of each member of (3). Just 13 of the 45 sets of three coplanar lines are concurrent.

Second, let $l = 1$. As above, we may set $g = i = 0$. Since

$$(zw) : (gi)(km); \quad z' = z + w : i' = i + g, \quad m' = m + k,$$

we may set $k = 0$, $m = 1$, unless $k = m = 0$. In the latter case, we replace x by $x + y$ and get a form free of y. In the former case, we apply the product of the above transformations $x' = x + z$, $y' = y + z$, and obtain a form having $g = i = k = 0$, $m = l = 1$, as before, but $p + 1$ in place of p. Hence we may take $p = 0$. Replacing x by $x + y$, we get

$$x^3 + xz^2 + xw^2 + y^2w + yw^2 + xzw, \tag{4}$$

for which (1110) is the only singular point. It contains only ten straight lines, all real; three coplanar lines concur if and only if one of them belongs to the pair giving the complete intersection of (4) with $w = 0$. See end of § 8.

6. Let the surface contain only three real points and let the three be not collinear. They can be transformed into 1, 2, 3. Then a, b, c, C,

E, F, r are zero, d, A, B, D, s, t, v are unity. Permuting x and y if necessary, we have $l = 1$, whence

$$S = w^3 + x^2y + gx^2z + (g+1)xz^2 + j(x^2w + xw^2) + ky^2z + (k+1)yz^2 + m(y^2w + yw^2) + p(z^2w + zw^2) + xyw + xzw + yzw.$$

First, let $k = 0$. Interchanging x and z, we get a like S with $k' = 0$, $g' = g + 1$, so that we may set $g = 0$. Replacing x by $x + \alpha w$, y by $y + \beta w$, we get a like S with $j' = j + \beta$, $m' = m$, $p' = p + \alpha + \beta$. Hence we may set $j = p = 0$, and get

$$(5) \qquad w^3 + x^2y + xz^2 + yz^2 + m(y^2w + yw^2) + xyw + xzw + yzw,$$

with no singular point if $m = 1$, and 2 as the only one if $m = 0$.

Second, let $k = 1$. If $g = 1$, we apply (xzy) and have an S with $k = 0$, just considered. If $g = 0$, replace x by $x + pw$, y by $y + jw$, z by $z + mw$; we get

$$(6) \qquad w^3 + x^2y + xz^2 + y^2z + xyw + xzw + yzw,$$

with just three singular points: $(z^2, z + z^2, z, 1)$, $z^3 + z + 1 = 0$.

Evidently there is no real line on these surfaces.

7. Let the three collinear points 1, 2, 5 be the only real points on the surface. Then c, d, F are unity and the others zero, so that

$$S = z^3 + w^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + k(y^2z + yz^2) + m(y^2w + yw^2) + pz^2w + (p+1)zw^2.$$

We make use of the transformations

$$(K) \quad x' = x + \alpha z + \beta w : k' = k + \alpha l, \quad m' = m + \beta l, \quad p' = p + \alpha j + \beta g;$$

$$(L) \quad y' = y + \gamma z + \delta w : \quad g' = g + \gamma l, \quad j' = j + \delta l, \quad p' = p + \gamma m + \delta k;$$

$$(M) \quad z' = z + \epsilon w : \quad j' = j + \epsilon g, \quad m' = m + \epsilon k, \quad p' = p + \epsilon.$$

If $l = 1$, we make $g = j = 0$ by L, $k = m = 0$ by K, $p = 0$ by M, and get

$$(7) \qquad z^3 + w^3 + zw^2 + x^2y + xy^2.$$

An evident imaginary transformation on z, w replaces (7) by (1).

Next, let $l = 0$. If g, j, k, m are all zero, we make $p = 0$ by M and have the binary form $z^3 + w^3 + zw^2$. In the contrary case, we permute the variables and have $g = 1$, noting that

$$(xy) : (gk)(jm); \quad (zw) : (gj)(km)(p, p+1).$$

Then we make $j = 0$ by M, $p = 0$ by K, $k = 0$ by use of $x' = x + y$. Then if $m = 0$, S is free of y; while, if $m = 1$, we have

(8) $$z^3 + w^3 + x^2z + xz^2 + y^2w + yw^2 + zw^2.$$

Neither (7) nor (8) has a singular point. They are not equivalent since the line $z = w = 0$ through our three real points is on (8) but not on (7).

8. Let the only real points on the surface be five coplanar points. They can be transformed into the points 1, 2, 3, 5, 6, viz., all but two of $w = 0$. Then d, D, v are unity and the others zero, so that

$$S = w^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + ky^2z + (k + 1)yz^2 + m(y^2w + yw^2) + p(z^2w + zw^2) + yzw.$$

We make use of the transformation

$$(yz) : (lg)(k, k + 1)(m, p),$$

which interchanges the two lines 125, 136, and the following transformations, which leave each line fixed:

(α) $x' = x + z$: $k' = k + l$, $p' = p + j$;
(β) $x' = x + w$: $m' = m + l$, $p' = p + g$;
(γ) $y' = y + w$: $j' = j + l$, $p' = p + k + 1$;
(δ) $z' = z + w$: $j' = j + g$, $m' = m + k$;
(ϵ) $x' = x + y$: $k' = k + g$, $m' = m + j$.

First, let g and l be not both zero. By use of (yz), we may set $l = 1$. Then by α, β, γ, we may set $k = m = j = 0$. If $g = 1$, we apply the product $\delta\gamma$ and have $k = m = j = 0$, $g = l = 1$, $p' = p + 1$, so that we may set $p = 0$ and get

(9) $$w^3 + x^2y + xy^2 + x^2z + xz^2 + yz^2 + yzw,$$

with no singular point. If $g = 0$, $p = 0$ or 1, we have

(10) $$w^3 + x^2y + xy^2 + yz^2 + yzw,$$

(11) $$w^3 + x^2y + xy^2 + yz^2 + z^2w + zw^2 + yzw,$$

the first having 6 as the only singular point, and the second having none. There are only three lines on (10): $w = y = 0$ and $y = w$, $z = x + bw$, $b^2 + b + 1 = 0$.

Second, let $g = l = 0$. By use of (yz), we may make $k = 0$. By use of γ, we may make $p = 0$. If $j = 0$, S is free of x. If $j = 1$, we use ϵ to make $m = 0$ and get

(12) $$w^3 + x^2w + xw^2 + yz^2 + yzw,$$

having 2 as the only singular point.

It remains to prove that (9) and (11), with no singular point, are not equivalent; likewise for (10) and (12), with a single singular point. Of our five real points, 1, 2, 5 and 1, 3, 6 alone are collinear. The line $z = w = 0$ through 1, 2, 5 is on (12), but not on (9), (10) or (11). The line $y = w = 0$ through 1, 3, 6 is on (10), (11), (12), but not on (9). There are only ten lines on (12), of which the two just given are the only real ones. We can derive (12) from (4) by the imaginary transformation replacing x by y, z by $z + y$, and y by $x + y + lw$, where $l^2 + l + 1 = 0$.

9. If five real points are not coplanar, four of them can be transformed into 1, 2, 3, 4, and, by permuting the variables, the fifth can be transformed into $k = 5$, 11 or 15. Of the resulting sets $S_k = [1, 2, 3, 4, k]$, no four points of S_{15} are coplanar, while 1, 2, 3, 11 is the only coplanar set in S_{11}. In S_{11} and S_{15} there is no set of three collinear points, while in S_5 the only such set is 1, 2, 5.

An automorph of the set S_5 must permute the collinear points 1, 2, 5 and hence also the remaining points 3, 4. The only transformations leaving fixed 3, 4 and the line $z = w = 0$ (of 1, 2, 5) are the binary transformations on x, y only. Next, 3 and 4 are interchanged by (zw), which leaves 1, 2, 5 fixed.

Let the points of set S_5 be the only real points on the cubic surface. Then a, b, c, d, A, t, v are zero, and the others unity. Thus

$$S = l(x^2y + xy^2) + gx^2z + (g+1)xz^2 + ix^2w + (i+1)xw^2 + ky^2z + (k+1)yz^2 + my^2w + (m+1)yw^2 + pz^2w + (p+1)zw^2 + xyz + xyw.$$

The automorphs of our set of fine points are generated by

$$(xy) : (gk)(im); \qquad (zw) : (gi)(km)(p, p+1);$$

$$x' = x + y : \quad k' = k + g + 1, \qquad m' = m + i + 1;$$

$$y' = y + x : \quad g' = g + k + 1, \qquad i' = i + m + 1.$$

First, let g, k, m, i be not all unity. Since they are permuted transitively by the above interchanges of variables, we may set $g = 0$. Then by use of $x' = x + y$ we may set $k = 1$. If $i = 0$, $m = 1$, we make $p = 0$ by use of (zw), and get

$$(13) \quad l(x^2y + xy^2) + xz^2 + xw^2 + y^2z + y^2w + zw^2 + xyz + xyw,$$

with no singular point if $l = 1$ and the only singular points 1 and 5 if $l = 0$. If $m = 0$, we make $i = 0$ by use of $y' = y + x$. By use of the product of $x' = x + y$ by (zw), we obtain a like form with $p' = p + 1$ and hence make $p = 0$ and get

$$(14) \quad l(x^2y + xy^2) + xz^2 + xw^2 + y^2z + yw^2 + zw^2 + xyz + xyw,$$

with the singular point 1 if $l = 0$, none if $l = 1$. If $i = m = 1$, we have

$$(15) \quad l(x^2y + xy^2) + xz^2 + x^2w + y^2z + y^2w + pz^2w + (p+1)zw^2 + xyz + xyw,$$

with the singular point 4 if $p = 1$, none if $p = 0$.

Second, let $g = k = m = i = 1$. We make $p = 0$ by use of (zw) and get

$$(16) \quad l(x^2y + xy^2) + x^2z + x^2w + y^2z + y^2w + zw^2 + xyz + xyw,$$

with the singular point 3 if $l = 1$, and 3, $(1y00)$, $y^2 + y + 1 = 0$, if $l = 0$.

The line $z = w = 0$ of the only three collinear points 1, 2, 5 is on any of our surfaces having $l = 0$, and not on those with $l = 1$. For $l = 0$, (14) and (15), with $p = 1$, are not equivalent, since their singular points are 1 and 4 respectively. For $l = 1$, (16) and (15), with $p = 1$, are not equivalent since their real non-singular points other than 1, 2, 5 are 4 and 3 respectively, and the tangent to the first at 4 is $z = 0$ which contains 1, 2, 4, 5, but not 3, while the tangent to the second at 3 is $w = x$ which contains 2 and 3, but not 1, 4 or 5.

In view of these facts and the number of singular points, it remains only to prove that, for $l = 1$, (13), (14), (15), for $p = 0$, are not equivalent, these being the only cases in which $l = 1$ and there is no singular point. The tangent plane to (15) at 3 is $x = 0$, which contains only 2, 3, 4 of our five real points; that at 4 is $z = 0$, which contains only 1, 2, 4, 5. The tangent plane to (13) at 4 is $z = x$, containing only 2 and 4; that to (14) at 4 is $x + y + z = 0$, containing only 4 and 5. The tangent plane to (13) or (14) at 3 is $x = 0$. Thus the tangent planes to (13) at 3 and 4 intersect in the line $x = z = 0$, containing only 2 and 4; while those to (14) intersect in $x = 0$, $y = z$, containing only 4.

10. Let 1, 2, 3, 4, 15 be the only real points on the surface. Then A, B, C, D, E, F are unity and the others zero. Permuting x, y and z, w, we may set $l = p = 0$. Then $(xz)(yw)$ replaces S by a like function with $g' = g + 1$. Hence we may set $g = 0$ and get

$$S = xy^2 + xz^2 + ix^2w + (i+1)xw^2 + ky^2z + (k+1)yz^2 + my^2w + (m+1)yw^2 + zw^2.$$

If $m = 0$, we interchange y with z and obtain a like S with $m' = 0$, $k' = k + 1$, so that we may set $k = 0$ and get

$$(17) \quad xy^2 + xz^2 + ix^2w + (i+1)xw^2 + yz^2 + yw^2 + zw^2,$$

with the single singular point 1 if $i = 0$, none if $i = 1$.

For $i = 0$, we replace y by $y + z$ and then x by $x + y$, w by $w + y$, and get

$$xw^2 = y^3 + yz^2 + z^3. \tag{17'}$$

The only lines on this surface are the six in $x = 0$ or $w = 0$. Although the partial derivative with respect to w is identically congruent to zero, the surface is not a cone.

If $m = 1$, we add x to y, z, w in S and get

$$k(xy^2 + xz^2 + y^2z) + (k + 1)(x^2z + x^2y + yz^2) + ixw^2 + (i + 1)x^2w + y^2w + zw^2.$$

First, let $k = 0$ and interchange x and y. If $i = 1$, we interchange y and z and get (17) for $i = 1$. If $i = 0$, we have S with $m = i = k = 1$, treated next. Second, let $k = 1$; we have S with $m = k = 1$, $i' = i + 1$. Hence we may set $i = 0$. Applying (ywz), we get (17) for $i = 0$.

11. Let 1, 2, 3, 4, 11 be the only real points on the surface. Then a, b, c, d, s, t, v are zero and the others unity. Interchanging x and y, we may set $l = 1$. Then $x' = x + z$, $y' = y + z$ replaces S by a like form with $g' = g + 1$. Hence we set $g = 1$. Applying (yz), we may set also $k = 1$. We get

$$x^2y + x^2z + ix^2w + (i + 1)xw^2 + y^2z + my^2w + (m + 1)yw^2 + pz^2w + (p + 1)zw^2 + xyz. \tag{18}$$

Denote it by $[imp]$. The cases in which there is no singular point are [101], [001], [011]; the second is transformed into the first by $x' = x + y$, $y' = x$, $z' = x + z$, and the third into the first by $x' = y$, $y' = x + y$, $z' = y + z$. For [111] there is the single singular point 4, which is invariant. For [100], [110] and [010], the only singular point is 3, and no two of the surfaces are equivalent since an automorph of our set of five points leaving 3 fixed is the identity or (xy). Finally, for [000] the partial derivative with respect to w is zero identically; replacing z by $x + y + z$ and w by $w + x + y$, we get

$$x^3 + x^2y + y^3 + xyz + zw^2.$$

It contains only nine straight lines, all imaginary,

$$w = \frac{1}{l^2}x + \frac{1}{l}y, \quad z = l^4x + l^2y; \qquad y = lx, \quad z = 0;$$

$$y = lx, \quad w = (l^2 + l)x \qquad (l^3 + l + 1 = 0).$$

The only singular points are 3 and $(x\,1\,x^2\,x^2 + x + 1)$, $x^3 + x^2 + 1 = 0$.

12. Let the points 1, 2, 3, 5, 6, 8, 11 of the plane $w = 0$ be the only real points on the surface. Then d is the only letter not zero, so that

$$S = w^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + k(y^2z + yz^2) + m(y^2w + yw^2) + p(z^2w + zw^2).$$

We employ the transformations (leaving w fixed)

(α) $\quad x' = x + w : \quad m' = m + l, \quad p' = p + g;$

(β) $\quad x' = x + y : \quad k' = k + g, \quad m' = m + j;$

(γ) $\quad x' = x + z : \quad k' = k + l, \quad p' = p + j;$

(δ) $\quad y' = y + z : \quad g' = g + l, \quad p' = p + m;$

(ϵ) $\quad y' = y + w : \quad j' = j + l, \quad p' = p + k.$

First, let l, g, k be not all zero. In view of

$$(xy) : (gk)(jm); \qquad (yz) : (lg)(mp),$$

we may set $l = 1$. We make $m = k = g = j = 0$ by use of α, γ, δ, ϵ. If $p = 0$, S is free of z. If $p = 1$, we have

$$xy(x + y) = w(w^2 + wz + z^2). \tag{19}$$

Since its real straight lines must lie in $w = 0$, there are just three. The surface is obtained from (1) by replacing z by $z + bw$, w by $z + b^2w$, where $b^2 + b + 1 = 0$.

Second, let $l = g = k = 0$. If, j, m, p are all zero, $S = w^3$. In the contrary case, we may set $j = 1$ and make $m = p = 0$ by β, γ. Then S is binary.

13. Let the only real points on the surface be six coplanar points (in $w = 0$) and a point $(abc1)$ not in $w = 0$. Adding aw, bw, cw to x, y, z, we have the plane $w = 0$ and point 4. The missing point in $w = 0$ can be transformed into 11, without altering w. Since 1, 2, 3, 4, 5, 6, 8 are the only real points on the surface, a, b, c, d, A, B, D are zero and the others are unity. Thus

$$S = l(x^2y + xy^2) + g(x^2z + xz^2) + ix^2w + (i + 1)xw^2 + k(y^2z + yz^2) + my^2w + (m + 1)yw^2 + pz^2w + (p + 1)zw^2 + \sigma,$$

$$\sigma \equiv xyz + xyw + xzw + yzw.$$

We may assume that g, k, l are not all zero in view of

(α) $\quad x' = x + y + z : \quad k' = k + l + g + 1, \quad m' = m + i + 1, \quad p' = p + i + 1.$

Then we may take $l = 1$ since

$$(xy) : (gk)(im); \qquad (xz) \sim (lk)(ip).$$

We may set $g = 0$. For, if $g = 1$, we may make $k = 0$ by α and then apply (xy).

First, let $i = 0$. Making $p = 0$ by use of α, we have

$$(20)\quad x^2y + xy^2 + xw^2 + k(y^2z + yz^2) + my^2w + (m+1)yw^2 + zw^2 + \sigma.$$

There is no singular point if $k = 1$. For $m = 1$, $k = 0$, the only singular points are 3 and 8; for $m = k = 0$, the only one is 3. The tangent plane at 4 is $x + (m+1)y + z = 0$; it contains 11 if and only if $m = 1$. Hence no two of the four types (20) are equivalent. The collinear triples determine the lines $z = w = 0$, $y = w = 0$, $x = w = 0$, $x + y + z = w = 0$; the first is not on (20), the second is, the third and fourth are on if and only if $k = 0$.

Second, let $i = 1$. If $k = 1$, $p = 0$, we apply (xz) and get an S with $l = 1$, $g = i = 0$, treated above. If $k = p = 1$, we have

$$(21)\quad x^2y + xy^2 + x^2w + y^2z + yz^2 + my^2w + (m+1)yw^2 + z^2w + \sigma.$$

Its singular points are $(x010)$, where $x^2 + x + 1 = 0$, and, in case $m = 1$, also 4. Thus no type (21) is equivalent to a type (20). Next, let $k = m = 0$; applying (xy), we have an S with $l = 1$, $g = i = 0$, considered above. Finally, if $k = 0$, $m = 1$, we have

$$(22)\qquad x^2y + xy^2 + x^2w + y^2w + pz^2w + (p+1)zw^2 + \sigma.$$

The only singular points are 3, 6, 8 if $p = 0$; the only one is 4 if $p = 1$. Hence no type (22) is equivalent to a type (20) or (21).

14. Consider five coplanar points and two points not in their plane. As in § 13, we may take the five to be 1, 2, 3, 5, 6, composed of the points in the two lines 125 and 136 in the plane $w = 0$, and the additional points to be 4, P. Now (yz) permutes 2 and 3, 5 and 6; $x' = x + y$ permutes 2 and 5, 8 and 11. Hence we may assume that the line joining 4 and P meets $w = 0$ at 1, 2 or 11. In the respective cases, $P = 7$, 9 or 15. The resulting three sets of seven points are considered in turn.

Let 1, 2, 3, 4, 5, 6, 7 be the only real points on the surface. Then D, E, F alone are unity, so that

$$S = l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + ky^2z + (k+1)yz^2 + my^2w + (m+1)yw^2 + pz^2w + (p+1)zw^2.$$

If l, g, j are all zero, there is no x. In the contrary case, we may set $g = 1$ by permuting y, z, w. We make $p = 0$, $k = 1$, by use of

$$x' = x + z: \quad k' = k + l, \quad p' = p + g;$$
$$x' = x + y: \quad k' = k + g, \quad m' = m + j.$$

Since (yw) now merely permutes l, j and adds 1 to m, we may set $m = 0$. We get

$$(23) \quad S_{lj} = l(x^2y + xy^2) + x^2z + xz^2 + j(x^2w + xw^2) + y^2z + yw^2 + zw^2.$$

If $j = 1$ there is no singular point. If $j = 0$, the only singular point is 5 if $l = 0$ and 7 if $l = 1$. There are just three collinear triples of our seven real points: 1, 2, 5 on $z = w = 0$; 1, 3, 6 on $y = w = 0$; 1, 4, 7 on $y = z = 0$. The first line is on (23) if and only if $l = 0$; the second is not; the third is if and only if $j = 0$. Hence no two of the four types (23) are equivalent.

For $j = 0$, the partial derivative of (23) with respect to w is zero identically. But (23) is then not a cone. For, if so, its vertex would be 5 or 7, according as $l = 0$ or 1; but 3, 5, 11 are collinear and also 2, 7, 12, while neither 11 nor 12 is on the surface.

15. Let 1, 2, 3, 4, 5, 6, 9 be the only real points on the surface. Then C, D, F, t, v are unity and the others zero, so that

$$S = l(x^2y + xy^2) + g(x^2z + xz^2) + ix^2w + (i + 1)xw^2 + ky^2z$$
$$+ (k + 1)yz^2 + m(y^2w + yw^2) + pz^2w + (p + 1)zw^2 + xzw + yzw.$$

The collinear triples are 1, 2, 5 (on $z = w = 0$); 1, 3, 6 (on $y = w = 0$); 2, 4, 9 (on $x = z = 0$). These lines are on S if and only if $l = 0$, $g = 0$, $m = 0$, respectively.

First, let $l = 1$. We may make $i = k = 0$ by using

$$y' = y + w: \quad i' = i + l, \quad p' = p + k + 1;$$
$$x' = x + z: \quad k' = k + l, \quad p' = p + i + 1,$$

and then make $p = 0$ by use of $(xy)(zw) : (gm)(ki)(p, p + 1)$. We get

$$(24) \quad x^2y + xy^2 + g(x^2z + xz^2) + xw^2 + yz^2 + m(y^2w + yw^2) + zw^2 + xzw + yzw,$$

with the one singular point 9 if $m = 0$, none if $m = 1$.

Second, let $l = 0$, $k \neq i$. We may make $k = 1$, $i = 0$, by use of $(xy)(zw)$, and then $p = 0$ by $x' = x + z$. We get

$$(25) \quad g(x^2z + xz^2) + xw^2 + y^2z + m(y^2w + yw^2) + zw^2 + xzw + yzw.$$

The only singular points are 1 and 3 if $g \neq 0$; 5 if $g = 1$, $m = 0$; none if $g = m = 1$. The four partial derivatives are linearly dependent only if $g = 0$, $m = 1$; but neither 1 nor 3 is an apex in view of 1, 4, 7 and 3, 2, 8.

Third, let $l = 0$, $k = i = 0$. The case $g = 1$, $m = 0$ may be dropped in view of $(xy)(zw)$. By using $y' = y + w$, we may set $p = 0$. Hence we have

$$(26)\quad g(x^2z + xz^2) + xw^2 + yz^2 + m(y^2w + yw^2) + zw^2 + xzw + yzw, \qquad (g, m) \neq (1, 0).$$

The four partial derivatives are linearly dependent only when $g = m = 1$; but the surface is then not a cone since no one of the four singular points

$$P = \left(y^2 + y,\ y,\ 1,\ \frac{y^2}{1 + y}\right), \qquad y^4 + y^3 + 1 = 0,$$

is a vertex (the point collinear with P and 3 not being on the surface). For $m = 1$, $g = 0$, the only singular point is 1. For $m = g = 0$, we replace y by $y + x$, w by $w + z$, and then x by $x + z$, and get $xw^2 + z^3 + yzw$, with the infinitude of singular points $(xy00)$. The plane $w = 0$ meets the surface in the triple line $w = z = 0$ (containing the singular points), which intersects the remaining lines $z = bw$, $x = by + b^3w$ on this ruled surface.

Fourth, let $l = 0$, $k = i = 1$. We make $p = 0$ by use of $(xy)(zw)$ and get

$$(27)\quad g(x^2z + xz^2) + x^2w + y^2z + m(y^2w + yw^2) + zw^2 + xzw + yzw.$$

The only singular points are 3 and 6 if $m = 1$, $g = 0$; 5 if $m = g = 1$; $P = (0y01)$, where $y^2 + y + 1 = 0$, if $m = 0$, $g = 1$; P, 3, 6 if $m = g = 0$. Only in the last case are the partial derivatives linearly dependent and the surface is not a cone since the third of the collinear points P, 1, $(1y01)$; 3, 2, 8; 6, 2, 11 are not on the surface.

That no two surfaces (24)–(27) are equivalent follows from the number and reality of the singular points, the fact that the pair 1, 2 is fixed by each automorph of our set of seven real points, so that 5 is fixed, and the fact that l is invariant, as well as the pair g, m (in view of the initial remark on the lines on S).

16. Let 1, 2, 3, 4, 5, 6, 15 be the only real points on the surface. Then C, D, E, F, s, t are unity and the others zero, so that

$$S = l(x^2y + xy^2) + g(x^2z + xz^2) + ix^2w + (i + 1)xw^2 + ky^2z + (k + 1)yz^2 + my^2w + (m + 1)yw^2 + pz^2w + (p + 1)zw^2 + xyw + xzw.$$

The only collinear triples are 1, 2, 5 (on $z = w = 0$) and 1, 3, 6 (on $y = w = 0$); the first line is on S if and only if $l = 0$, the second if and only if $g = 0$.

First, let $l = g = 0$. We make $k = 0$ and $m = 0$ by use of

$$(yz) : (lg)(mp)(k, k+1),$$

$$(\beta) \qquad x' = x + w, \quad y' = y + w, \quad z' = z + w : \quad i' = i + l + g,$$
$$m' = m + l + k + 1, \quad p' = p + g + k + 1.$$

We get

$$(28) \quad ix^2w + (i+1)xw^2 + yz^2 + yw^2 + pz^2w + (p+1)zw^2 + xyw + xzw.$$

The only singular points are 2, 5 if $i = 1$, and 1, 2 if $i = 0$.

Second, let $l = g = 1$. We make $k = 0$ by (yz). By use of

$$(\alpha) \qquad x' = x + y + z : \quad k' = k + l + g, \quad m' = m + i + 1,$$
$$p' = p + i + 1,$$

we make $m = 0$ if $i = 0$, and get

$$(29) \quad x^2y + xy^2 + x^2z + xz^2 + xw^2 + yz^2 + yw^2 + pz^2w + (p+1)zw^2 + xyw + xzw,$$

with no singular point. If $i = 1$, S becomes

$$(30) \quad x^2y + xy^2 + x^2z + xz^2 + x^2w + yz^2 + my^2w + (m+1)yw^2 + pz^2w + (p+1)zw^2 + xyw + xzw.$$

There is no singular point if $p = 0$; for $p = 1$, the only one is 15 if $m = 0$ and 4 if $m = 1$.

Finally, let $l = 1$, $g = 0$. We make $k = i = 0$ by α, β and get

$$(31) \quad x^2y + xy^2 + xw^2 + yz^2 + my^2w + (m+1)yw^2 + pz^2w + (p+1)zw^2 + xyw + xzw.$$

The only singular point is 6 if $p = 1$; 15 if $p = m = 0$; none if $p = 0$, $m = 1$.

Our three cases are distinguished by the invariance of the pair l, g. With each case there is no further normalization, as shown by a consideration of the eight products of (yz), α, β.

17. Consider a set of 7 points no 5 of which are coplanar. Then no 3 of the 7 are collinear. For, if 3 are in a line l, each of the remaining 4 points lies in a plane with l. But only 3 planes pass through a line l. Hence at least 2 of the 4 points are coplanar with l, whereas no 5 of the 7 points are coplanar.

Certain 4 of 6 points are coplanar since, otherwise, the 20 combinations of the 6 three at a time would give more than the 15 existing planes. Hence we may assume that one of our 7 points is 4 and that four of them

are in the plane $w = 0$. Three of the latter can be transformed into 1, 2, 3, without altering 4 or $w = 0$. The only point of $w = 0$ not collinear with two of 1, 2, 3 is 11. Thus five of our points are 1, 2, 3, 4, 11. The only points not collinear with two of these five and not in $w = 0$ are 12, 13, 14. The latter are permuted transitively by the permutations of x, y, z (which do not alter our set of five points). Hence we can transform any set of 7 points no five coplanar into 1, 2, 3, 4, 11, 12, 13, so that all such sets are equivalent.

Since 3 of 5 points are collinear, an equivalent definition of such a set of 7 points is that no 3 of its points are collinear. Hence the set can be transformed into 7, 9, 10, 12, 13, 14, 15. Let these be the only real points on the cubic surface. Then, C, E, F, s, t, v are zero and the others unity. By use of (xy), $x' = x + z$, and (yz), we may set $l = g = k = 0$, and get

$$(32)\qquad \begin{aligned}[i, m, p] = {} & x^3 + y^3 + z^3 + w^3 + xy^2 + xz^2 + i(x^2w + xw^2) \\ & + yz^2 + m(y^2w + yw^2) + p(z^2w + zw^2) + xyz.\end{aligned}$$

Replacing y by $y + z$ and z by y, we get $[i, m + p, m]$. In $[i, m, p]$, we replace x by $x + y + z$ and then interchange x and z; we get $[i + p, m + i, i]$. Hence $[i, m, p]$ is equivalent to [001] or [000]. The only singular points of the latter are $(y^2 + 1, y, 1, 0)$, where $y^3 + y + 1 = 0$; the only one of the former is 13.

18. Any set of six real points is equivalent to one of

$$S_k = [1, 2, 3, 4, 5, k], \quad k = 6, 10, 13; \qquad S = [1, 2, 3, 4, 11, 12],$$

$$\Sigma = [1, 2, 3, 5, 6, 8].$$

For, if the six are coplanar, they can be transformed into the points Σ other than 11 of $w = 0$. If only five are coplanar the set can be transformed into S_6, 4 being the only point not in $w = 0$. The case in which no four of the six points are coplanar was excluded in § 17.

It remains to discuss the case in which four, but not more than four, of the six points are coplanar. We may assume that four of the points are in $w = 0$ and that a fifth point is 4. If no three of the six points are collinear, the argument in § 17 shows that the set is equivalent to S. If three are collinear they may be taken to be 1, 2, 5 and 3 to be the fourth point in $w = 0$. The line joining 4 and the 6th point P of the set meets $w = 0$ at a point Q not 1, 2, or 5 (since five points are not coplanar); if $Q = 3$, then $P = 10$ and the set is S_{10}; if $Q = 8$ or 11, we apply (x, y) or $y' = y + x$, neither of which alters 3, 4 or the line 1, 2, 5, and obtain $Q = 6$, whence $P = 13$, and the set is S_{13}.

We are now ready to discuss surfaces containing exactly nine real

points. First, let the only real points not on the surface be the six in set Σ. Then $d = r = 0$ and the others are unity. We make $l = 0$ by (xy). First, let $k = 1$. We make $g = i = m = p = 0$ by use of (xz), $x' = x + w$, and $y' = y + w$, and $y' = y + w$, $z' = z + w$, obtaining

$$x^3 + y^3 + z^3 + xy^2 + xz^2 + xw^2 + y^2z + yw^2 + zw^2 + xyw + xzw + yzw. \tag{33}$$

The only singular point is 7. Second, let $k = 0$. If $g = 0$, we apply $x' = x + y + z$ and then (xy) and get an S with $l = g = 0$, $k = 1$, just treated. If $g = 1$, we make $i = 0$ by $x' = x + w$ and $m = 0$ by $y' = y + w$. Then if $p = 1$, we replace x by $x + z$, y by $y + z$ and get a form free of z. If $p = 0$, we have

$$x^3 + y^3 + z^3 + xy^2 + x^2z + xw^2 + yz^2 + yw^2 + zw^2 + xyw + xzw + yzw, \tag{34}$$

with no singular point and no real straight line.

19. Let 1, 2, 3, 4, 5, 6 be the only real points not on the surface. Then C, D, E, F, s, t are zero and the others unity. By use of the automorphs $x' = x + ay + bz$ and their products by (yz), we may set $l = g = 0$ and discard the case $m = 1$, $p = 0$, obtaining

$$x^3 + y^3 + z^3 + w^3 + xy^2 + xz^2 + j(x^2w + xw^2) + k(y^2z + yz^2) + m(y^2w + yw^2) + p(z^2w + zw^2) + xyz + yzw, \quad (m, p) \neq (1, 0). \tag{35}$$

No two of these types are equivalent. If $m = p = k = 0$, $j = 1$, we replace w by $w + x + y$, y by $y + z$ and get $xy^2 + w(y^2 + yz + z^2 + wy + w^2)$. Replacing x by $x + w$, z by $z + w$, we get $xy^2 + wz(z + y)$, a ruled surface with the directrices $x = w = 0$, $y = z = 0$ and ruled lines $y = \alpha z$, $w = \alpha^2 x/(\alpha + 1)$; $x = z = 0$; $x = 0$, $y = z$. Its singular points are $(x00w)$. For the remaining cases, the only singular points are

for $m = p = 0$: 15 if $k = 1$, $j = 0$; 7 if $k = j = 1$; 15, $(01z\,z + 1)$, $z^2 + z + 1 = 0$, if $k = j = 0$;

for $m = p = 1$: 15 if $j = k = 0$; 12, 13 if $j = 0$, $k = 1$; 9, 10, 11 if $j = 1$, $k = 0$; 14, $(w\,w + 1\,1\,w)$, $w^2 + w + 1 = 0$, if $j = k = 1$.

for $m = 0$, $p = 1$: 10 if $j = k = 1$; 11, 13 if $j = k = 0$; none if $j \neq k$.

The collinear triples are 7, 8, 15 on $y = z$, $x = w$; 7, 11, 14 on $y = z$, $x = z + w$; 8, 9, 10 on $x = 0$, $y = z + w$; 8, 12, 13 on $x = w$, $y = z + w$; 9, 11, 13 on $x = z$, $y = z + w$; 10, 11, 12 on $x = y$, $z = y + w$. For $m = 0$, $p = 1$, the first line is not on (35), the others are on if $j = 0$, $k = 1$,

$k = 0$, $k = j$, $k = j$, respectively; in particular, with just two real lines if $j = 0$, $k = 1$.

20. Let 1, 2, 3, 4, 5, 10 be the only real points not on the surface. Then a, b, c, d, A, F are unity and the others zero. We make $l = p = 0$ by use of (xy) and (zw); and get

$$[gjkm] = x^3 + y^3 + z^3 + w^3 + xy^2 + g(x^2z + xz^2) + j(x^2w + xw^2) + k(y^2z + yz^2) + m(y^2w + yw^2) + zw^2. \tag{36}$$

The only transformations permuting forms of this type are generated by

$$x' = y, \quad y' = x + y; \quad (xz)(yw); \quad z' = w, \quad w' = z + w.$$

Under these, [0000] is invariant; [1001], [0111], [1110] are permuted; [0110], [1011], [1101] are permuted; while the remaining nine are permuted. Hence there are four non-equivalent types (36). Type [1000] has the one singular point 6, dependent derivatives, but is not a cone. Type [1001] has the singular points 6 and $(x\,1\,x + 1\,1)$, x arbitrary; by the transformation

$$X = x + y + w, \quad Y = y + w, \quad Z = x + z, \quad W = x + z + w,$$

[1001] equals $XY^2 + ZW^2$, with the singular points $(X0Z0)$; it is a ruled surface with directrices $x = z = 0$, $y = w = 0$, and ruled lines $z = d^2x$, $y = dw$, and $x = w = 0$. Type [0110] has no singular point and the single real line $z = x$, $y = w$. Type [0000] has no singular point; the only three real lines on it are $z = x$, $y = w$; $y = z = x + w$; $w = x$, $z = x + y$; an evident imaginary transformation, cogredient in x, y and z, w, transforms it into (1).

21. Let 1, 2, 3, 4, 5, 13 be the only real points not on the surface. Then a, b, c, d, A, v are unity and the others zero. By use of $x' = x + y$, we make $l = 0$ and get

$$x^3 + y^3 + z^3 + w^3 + xy^2 + g(x^2z + xz^2) + j(x^2w + xw^2) + k(y^2z + yz^2) + m(y^2w + yw^2) + p(z^2w + zw^2) + yzw. \tag{37}$$

First, let $g = j = 1$. If $k = m = 1$, the only singular points are 8, 9, 15 if $p = 0$, and $(10w^2w)$, where $w^2 + w + 1 = 0$, if $p = 1$. If $km = 0$, we make $m = 0$ by (zw), and $k = 0$ by

$$(\alpha) \quad x' = x + z, \quad w' = w + z: \quad g' = g + j + 1, \quad k' = k + m + 1, \quad p' = p + j + 1.$$

Then the singular points are 6, 7, 15 if $p = 1$ and $(0yy^21)$, where $y^2 + y + 1 = 0$, if $p = 0$.

Second, let $gj = 0$. We make $j = 0$ by (zw) and then $g = 0$ by α. In view of (zw), we may interchange k and m. For $k = m = 0$, the only singular points are (yyy^21), $y^3 = 1$, if $p = 0$, and none if $p = 1$. For $k = 1$, $m = 0$, the only singular points are 10 and 11 if $p = 1$, none if $p = 0$. For $k = m = 1$, the only singular points are 11 and 12 if $p = 0$, 15 if $p = 1$.

The resulting types (37) having the same number of real and imaginary singular points are seen to be not equivalent either by the incidence of these points with the invariant plane $y = 0$ or by the intersections of the latter with the surface.

The types (37) with no singular point have $j = g = m = 0$, and either $k = 0$, $p = 1$ or $k = 1$, $p = 0$. For the former the three real lines on the surface are $y = z$, $x + w = 0$ or z, and $y = w$, $x = z + w$. The last is the only real line in the second case.

22. Let 3, 6, 9, 10, 12, 13 be the only real points not on the surface. Since no three of the six points are collinear, the set is equivalent to S of § 18. Then c, D, E, v are unity and the others zero. We get

$$z^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + ky^2z + (k + 1)yz^2 + my^2w + (m + 1)yw^2 + p(z^2w + zw^2) + yzw. \tag{38}$$

We normalize by the transformations

$$x' = x + \alpha y + \beta z + \gamma z; \qquad y' = y + z + w, \quad z' = w, \quad w' = z;$$

and $w' = w + z$. First, let $l = 0$. If $g = j = 0$, x does not occur. In the contrary case, we may make $j = 1$, $g = p = m = 0$. Then the only singular point is 2 if $k = 0$, 11 if $k = 1$. Second, let $l = 1$. We may make $k = m = j = 0$. For $g = 1$, there is no singular point if $p = 0$ and the factor $y + z$ if $p = 1$. For $g = 0$, there is no singular point if $p = 1$, while the only one is 7 if $p = 0$. No two of the resulting types are equivalent. Those with no singular point are given by $l = 1$, $k = m = j = 0$, and $g = 1$, $p = 0$ or $g = 0$, $p = 1$; each has only three real straight lines: for the former, $y = z = 0$; $y = z$, $w = 0$; and $y = z = w$; for the latter, $y = z = 0$; $x = 0$, $y = z$; $x = y = z$.

23. Finally, consider the surfaces with exactly eleven real points. Here let 1, 2, 3, 5 be the only real points not on the surface. Then d, B, D, r, t, v are zero and the others unity. We make $l = i = m = 0$ by (xy), $x' = x + w$, and $y' = y + w$. By the product of $x' = x + y$, $y' = x$, and the last two transformations, we make $g = 0$, and get

$$x^3 + y^3 + z^3 + xy^2 + xw^2 + k(y^2z + yz^2) + yw^2 + pz^2w + (p + 1)zw^2 + xyw. \tag{39}$$

If $k = 0$, we make $p = 0$ by $z' = z + w$; then the only singular point is 13. For $k = 1$, the only singular points are 7, 11, 15 if $p = 1$; $(0, y, y + 1, 1)$ where $y^2 + y + 1 = 0$, if $p = 0$.

24. Let 1, 2, 3, 11 be the only real points not on the surface. Then a, b, c, C, E, F are unity and the others zero. Thus

$$S = x^3 + y^3 + z^3 + l(x^2y + xy^2) + g(x^2z + xz^2) + (j + 1)x^2w + jxw^2 + k(y^2z + yz^2) + my^2w + (m + 1)yw^2 + pz^2w + (p + 1)zw^2.$$

If $lg = 0$, we make $l = g = 0$ by $(yz) : (lg)(mp)$ and

$$(\alpha) \quad x' = x + z, \quad y' = y + z : \quad g' = g + l + 1, \quad k' = k + l + 1, \quad p' = p + j + m + 1.$$

We make $j = m = 0$ by $x' = x + w$ and $y' = y + w$. If $k = 0$, we make $p = 0$ by $z' = z + w$ and get

$$x^3 + y^3 + z^3 + x^2w + yw^2 + zw^2, \tag{40}$$

with the one singular point 14, not a vertex by 14, 15, 1. Next, let $k = 1$. If $p = 0$, we replace y by $y + z$ and obtain a form free of z. If $p = 1$, we have

$$x^3 + y^3 + z^3 + x^2w + y^2z + yz^2 + yw^2 + z^2w. \tag{41}$$

The latter has no singular point and contains exactly five real straight lines: $y = 0, z = x$; $y = w = x + z$; $z = 0, y = x + w$; $z = w, y = x$; $w = 0, x = y + z$.

Next, let $l = g = 1$. If $k = 0$, we apply $y' = y + x$, $z' = z + x$, and have $l = g = 0$, just treated. Thus let $k = 1$. Make $j = 0$ by $x' = x + w$. If $m = 0$, make $p = 0$ by α. If $m = 1$, $p = 0$, make $m = 0$ by (yz). Hence we may set $m = p$. Replacing x by $x + y + z$ and then y by $y + z$, we get a form free of z.

25. Lastly, let 11, 12, 13, 14 (in preference to 1, 2, 3, 4) be the only real points not on the surface. Then r, s, t, v alone are unity. We get

$$l(x^2y + xy^2) + g(x^2z + xz^2) + j(x^2w + xw^2) + k(y^2z + yz^2) + m(y^2w + yw^2) + p(z^2w + zw^2) + xyz + xyw + xzw + yzw. \tag{42}$$

The only available transformations are the permutations of x, y, z, w. The six coefficients $l, \cdots, p$ are permuted transitively since

$$(xy) : (gk)(jm); \quad (xz) : (lk)(jp); \quad (xw) : (lm)(gp).$$

If all six are zero, the only singular points are 1, 2, 3, 4, no one being a vertex. If five are zero, we may set $l = 1$; the singular points are 3, 4.

Next, let $l = 1$ and a single other one be unity; the surface with $g = 1$ is transformed into that with k, j or m unity by (xy), (zw), $(xy)(zw)$; in the remaining case $p = 1$, $x + y + z + w$ is a factor. For $l = g = 1$, $j = k = m = p = 0$, the only singular points are 4, 7.

If $l = g = j = 1$, $k = m = p = 0$, the only singular point is 15. By use of (xy), we obtain $l = k = m = 1$. Next, if $l = g = k = 1$ or $l = j = m = 1$ (and the remaining coefficients vanish), $x + y + z + w$ is a factor. The six remaining cases in which $l = 1$ and two further coefficients are unity are derived by use of (xy), (xw), (yz), (xyw), (xzy) from the case $l = g = m = 1$. For the latter, 10 is the only singular point.

If a single coefficient is zero, we may set $l = 0$. The only singular points are then $(x100)$, $x^2 + x + 1 = 0$. If only two coefficients are zero, we may set $l = p = 0$ or $l = g = 0$. In the first case, the only singular points are $P = (x100)$, $x^2 + x + 1 = 0$, and $Q = (00z1)$, $z^2 + z + 1 = 0$, no one being a vertex since P and 3 are collinear with $(x110)$, not on the surface. For $l = g = 0$, $j = k = m = p = 1$, there is no singular point and just five real lines: $z = w = 0$; $y = w = 0$; $w = 0$, $x = y + z$; $x = 0$, $y = z + w$; $y = z$, $x = w$, of which the first three are coplanar but not concurrent, while the first and last two are coplanar and concurrent.

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NORMAL FORMS FOR ONE- AND TWO-SIDED SURFACES.

By James W. Alexander, II.

1. In view of the difficulties that are still to be overcome before normal forms can be obtained for manifolds of more than two dimensions, a simplified treatment of the analogous problem for closed surfaces may be worth mentioning. We shall prove that all one-sided surfaces of a given connectivity k are topologically equivalent and can therefore be reduced to a single normal form; also, that the same is true of all closed surfaces of a given genus $p = (k - 1)/2$.* Wherever the reasoning is of an intuitional character, it may readily be put into perfectly rigorous form by means of known theorems. We shall assume that the connectivity of a surface, as defined by the generalized Euler relation

$$\alpha_0 - \alpha_1 + \alpha_2 = 3 - k,\dagger$$

is finite and invariant, and furthermore that the sphere is a normal form to which every closed surface of connectivity 1 may be reduced.

2. Upon any closed surface S, let us describe as many non-intersecting closed paths as possible without separating the surface into distinct regions. According to the assumptions made in § 1, the number of these paths is finite. Then if we cut the surface S along the paths, there will remain a surface S' bounded by certain curves which we shall suppose to be $k - 1$ in number. The points of these boundary curves will be included in the surface S' so that there will really be two points of S' to every point of a cut in S.

It can easily be shown that the surface S' is equivalent to a sphere from which $k - 1$ simply connected regions have been removed. For if we were to match the edge of a simply connected bounded surface along each of the bounding curves of S', we should obtain a closed surface S'' consisting of S' and the adjoined surfaces. Moreover, the surface S'' would surely be equivalent to a sphere, otherwise a closed cut could be made in S'' without severing it. But if this were possible, the cut could always be made in such a way as not to pass through the adjoined surfaces,

* A proof of this theorem is sketched in the article on Analysis Situs by Dehn and Heegaard in the Encyklopädie der mathematischen wissenschaften where references to the literature will be found.

† In this equation α_0, α_1 and α_2 represent the number of vertices, edges, and faces respectively of an arbitrary decomposition of the surface into simply connected pieces.

the latter being simply connected. We should thus be able to make a closed cut in S' without severing it, contrary to hypothesis. The surface S' will for convenience be regarded as a Riemann plane from which the interiors of $k - 1$ circles have been removed.

3. Now two classes of closed cuts may be distinguished on a surface. If we start at a point P on one edge of the cut and describe a path which follows along the cut without crossing it, we shall return, after having made a complete circuit, either to the point P or to the point P' homologous to P but on the other edge of the cut. When the first of the two possibilities is realized, the cut is said to be two-sided; in the contrary case, it is said to be one-sided. A two-sided cut gives rise to two boundary curves which are in point-for-point continuous correspondence with one another; a one-sided cut gives rise to a single curve only, consisting of two segments ABC and CDA which are in point-for-point continuous correspondence with one another.

4. Returning to the representation S' of S in the Riemann plane, we shall distinguish three types of bounding circles.

I. Circles arising from one-sided cuts, opposite points of which (i. e. points at the extremities of the same diameter) correspond to the same point of S.

II. Pairs of circles arising from two-sided cuts and such that when a point P be made to describe one of the cuts, the two images of P in S' always describe their respective circles in the same sense.

III. Pairs of circles arising from two-sided cuts and such that when a point P be made to describe one of the cuts, the two images of P in S' describe their respective circles in opposite senses.

5. When and only when the circles of the surface S' are all of type III, the surface S is two-sided. In this case, S' can readily be reduced to any one of the well-known normal forms. Thus, Klein's normal form, the sphere with p handles, is obtained by deforming the surface S' in such a way that each circle becomes the extremity of a tubular neck protruding from the body of the surface. By joining corresponding circles to one another, each pair of tubular necks will give rise to a handle.

Another convenient normal form for a two-sided surface other than a sphere is a plane bounded by p rectangles, where opposite points of a rectangle correspond to the same point of S. This form can be obtained at once if we cut the surface S' along a set of non-intersecting curves, one for each pair of circles, and such that each curve joins a point of one circle to the point on the other circle of the pair which represents the same point of S. For every curve of the new boundary can then be deformed into a rectangle, two of whose sides will correspond to a pair of circles, while the remaining two will correspond to the edges of a cut.

Finally, if we deform all of the rectangles so as to make a corner of each pass through the same point P, we shall have another well-known form, a simply connected surface bounded by a chain of arcs

$$a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}, \quad \cdots, \quad a_pb_pa_p^{-1}b_p^{-1}.*$$

which may be deformed into a polygon.

6. The model S' of a one-sided surface S contains at least one circle of Type I, or else a pair of circles of Type II. Let us first observe that

A pair of circles of Type II can be replaced by two circles of Type I.

Join the points P and Q of the first circle to the two corresponding points P' and Q' respectively of the second by means of two simple non-intersecting curves p and q respectively. Cut the surface S' along the lines p and q, thereby severing from S' a piece E bounded by one edge of the cut p, an arc r of the first circle, one edge of the cut q, and an arc s of the second circle (Fig. 1).

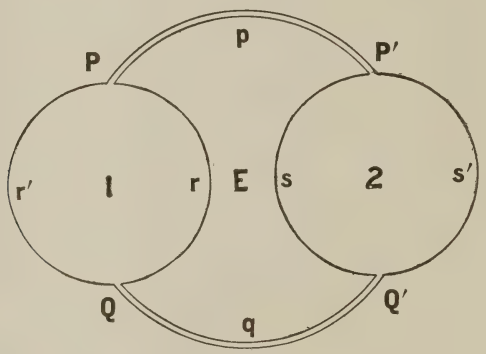


Fig. 1.

Now the arc r of the first circle measured from P to Q corresponds to the same points of the surface S as the arc s' complementary to s on the second circle measured from P' to Q'. Similarly, the arc s measured from P' to Q' corresponds to the same set of points as the arc r' complementary to r measured from P to Q. We may therefore re-affix the element E to the surface by matching s against r' and r against s'. The openings which corresponded to the circles are now closed up and there remain two other openings, one bounded by both edges of the cut q, the other by both edges of the cut p. The boundary of each of these openings is equivalent to a circle of Type I.

Furthermore,

* The symbols a_i and a_i^{-1} denote the same arc, but in one case, the arc is described in one sense, in the other case, in the other.

A pair of circles of Type III may be replaced by a pair of Type II, provided there exists at least one circle of Type I.

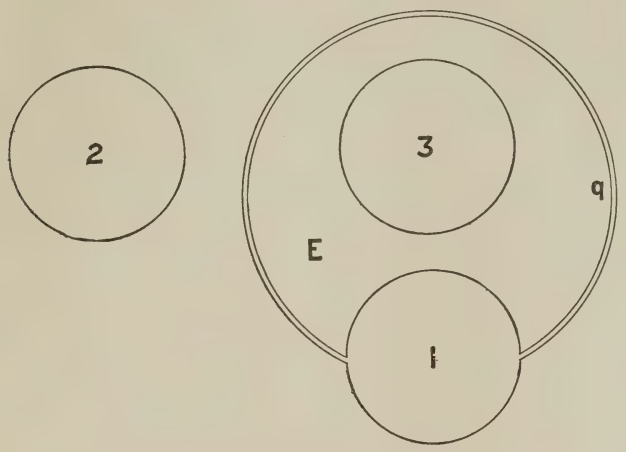


Fig. 2.

For let 1 be a circle of Type I, and 2 and 3, a pair of circles of Type III. If a cut q be made joining two opposite points of 1, a piece E of the surface will be severed from the rest. The surface E may be rejoined by matching the arc of the circle 1 belonging to E with the complementary arc, since the two arcs correspond to the same points of S. The circle 1 is thereby replaced by a curve composed of the two edges of q and equivalent to a circle of Type I. But the surface E must have been replaced with its lower and upper faces interchanged, since opposite points of the circle 1 corresponded to one another. Therefore, the sense of any closed curve traced upon E must have been reversed by the transformation. If then the cut q be chosen so that the surface E contains the circle 3 but not the circle 2, the pair will be transformed from one of Type III to one of Type II.

By means of the above two theorems, the diagram S' of a one-sided surface can always be reduced to a plane surface bounded by k–1 circles of Type I. Another normal form can be obtained by deforming the k–1 circles into curves passing through a common point P. Remembering that each circle may be subdivided into two arcs which correspond to the same arc of the surface S, we shall then have the one-sided surface S represented by a simply connected surface bounded by a chain of arcs

$$a_1 a_1 a_2 a_2 \cdots a_{k-1} a_{k-1}$$

which may be deformed into a polygon.

PRINCETON, N. J.

MINIMA OF DOUBLE INTEGRALS WITH RESPECT TO ONE-SIDED VARIATIONS.*

By Charles Albert Fischer.

In some problems in minima of double integrals the surface over which the integral is taken is restricted to lie in a given closed region R. Then it may happen that there is no extremal surface bounded by a previously given space curve which lies entirely in R, but that there is a surface bounded by the given curve, consisting of an extremal surface and a part of the boundary of R, which minimizes the given integral. In the first two sections of this paper some necessary conditions for such a minimum are derived, and in the last section there is a set of sufficient conditions.

The treatment of the analogous problem where the variations are unrestricted has been greatly simplified by Radon.† He shows that if the value of the double integral

$$\int\int \Phi\left(x, y, z, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) du dv$$

is to be unaffected by any change in the parametric representation of the surface over which the integral is taken, Φ must be expressible in the form

$$\Phi\left(x, y, z, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = F(x, y, z, A, B, C),$$

where

$$A = \frac{\partial(y, z)}{\partial(u, v)}, \qquad B = \frac{\partial(z, x)}{\partial(u, v)}, \qquad C = \frac{\partial(x, y)}{\partial(u, v)},$$

and then the proofs are much simpler. Consequently, the integral for which a minimum is sought will be given in the form

$$J = \int\int_{\Omega} F(x, y, z, A, B, C) du dv.$$

§ 1. The Analogue of the Lagrange Equation.

Suppose there is a surface of class D''‡ consisting of an extremal surface,

* Read before the American Mathematical Society, Apr. 24, 1915.

† Monatshefte für Mathematik und Physik, 22 (1911), p. 53.

‡ Bolza, Vorlesungen über Variationsrechnung, p. 664.

S: $\qquad x = x(u, v), \quad y = y(u, v), \qquad z = z(u, v),$

and a part of the boundary of R, whose equations are

$\tilde{S}$: $\qquad x = \tilde{x}(u, v), \qquad y = \tilde{y}(u, v), \qquad z = \tilde{z}(u, v),$

which gives as small a value to the integral J as any other surface of class D', belonging to R, that is in its neighborhood, and is bounded by the same space curve. The surface $\tilde{S}$ will be assumed to be of class C'', and its equations taken in such a way that $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ agree in sign with the direction cosines of the normal to $\tilde{S}$ directed into R. The intersection of S and $\tilde{S}$ will be called

L: $\qquad x = x(s), \quad y = y(s), \qquad z = z(s),$

where s is the length of arc.

A function $\omega(u, v)$ will be selected, which is of class D' in the neighborhood of $\tilde{S}$, vanishes along L, and is nowhere negative. Then if

$$\delta x = \epsilon\tilde{A}\omega(u, v), \qquad \delta y = \epsilon\tilde{B}\omega(u, v), \qquad \delta z = \epsilon\tilde{C}\omega(u, v),$$

the varied surface is entirely in R, for small positive values of ϵ, and the first variation of J becomes*

$$\delta J = \epsilon\int\int T\left(x, y, z, \frac{\partial x^2}{\partial u}, \cdots, \frac{\partial^2 z^2}{\partial v^2}\right)(\tilde{A}^2 + \tilde{B}^2 + \tilde{C}^2)w(u, v)dudv.$$

It follows that

$$T\left(x, y, z, \frac{\partial\tilde{x}^2}{\partial u}, \cdots, \frac{\partial\tilde{z}^2}{\partial v^2}\right) \geqq 0 \tag{1}$$

at every point of the part of the boundary of R which belongs to the minimizing surface.

§ 2. The Angle of Intersection of S and $\tilde{S}$.

It will next be proved that the surfaces S and $\tilde{S}$ must intersect at such an angle that the Weierstrass E-function,

$$E(x, y, z; A, B, C; \tilde{A}, \tilde{B}, \tilde{C}) \equiv F(x, y, z, \tilde{A}, \tilde{B}, \tilde{C})$$
$$- \tilde{A}F_A(x, y, z, A, B, C) - \tilde{B}F_B(x, y, z, A, B, C) - \tilde{C}F_C(x, y, z, A, B, C),$$

vanishes at every point of L.

Suppose that there is a point P where it does not vanish. Then there is a segment of L including P on which E has a permanent sign. Let $\omega(s)$ be a function of class C'' which agrees in sign with E in this segment and vanishes outside of it. The segment will be taken small enough to exclude any points where any of the derivatives $x'(s)$, $y'(s)$ and $z'(s)$ are

* Radon, loc. cit., p. 58, equation (15).

discontinuous. Then a part of $\tilde{S}$ including this segment of L can be represented by equations of the form

$$x = \tilde{x}(s, n), \quad y = \tilde{y}(s, n), \quad z = \tilde{z}(s, n),$$

where n is the length of a curve on $\tilde{S}$ normal to L. Since $E(x, y, z; A, B, C; \tilde{A}, \tilde{B}, \tilde{C})$ is homogeneous in $\tilde{A}, \tilde{B}, \tilde{C}$ such a change of parameters does not affect its non-vanishing. Three functions, $\xi(u, v)$, $\eta(u, v)$, $\zeta(u, v)$, will now be chosen in such a way that they vanish along the boundary of Ω, and their values along L are

$$\begin{aligned} \xi(u(s), v(s)) &= \tilde{x}_n(s, 0)\omega(s), \\ \eta(u(s), v(s)) &= \tilde{y}_n(s, 0)\omega(s), \\ \zeta(u(s), v(s)) &= \tilde{z}_n(s, 0)\omega(s). \end{aligned}$$

We take also the one-parameter family of surfaces defined by the equations

$$\bar{S}: \qquad \begin{aligned} \bar{x} &= x(u, v) + \epsilon\xi(u, v), \\ \bar{y} &= y(u, v) + \epsilon\eta(u, v), \\ \bar{z} &= z(u, v) + \epsilon\zeta(u, v). \end{aligned}$$

The increment ΔJ is then the difference between the value of the integral taken over $\bar{S}$ and a part of $\tilde{S}$, and its value taken over S and a slightly different part of $\tilde{S}$. The first variation is found to be*

$$\begin{aligned} \delta J = \epsilon \int\int_S T(A\xi + B\eta + C\zeta)dudv - \epsilon \int_L \begin{vmatrix} F_A & F_B & F_C \\ \tilde{x}_n & \tilde{y}_n & \tilde{z}_n \\ \tilde{x}_s & \tilde{y}_s & \tilde{z}_s \end{vmatrix} \omega(s)ds \\ + \epsilon \lim_{\epsilon=0} \left[\frac{1}{\epsilon} \int\int_{\Delta\bar{S}} F(x, y, z, \bar{A}, \bar{B}, \bar{C})dudv \right. \\ \left. - \frac{1}{\epsilon} \int_L \int_0^{N(s, \epsilon)} F(x, y, z, \tilde{A}, \tilde{B}, \tilde{C})dnds \right]. \end{aligned} \tag{2}$$

The first integral vanishes because S is an extremal surface. The region $\Delta\bar{S}$ is the part of $\bar{S}$ between its intersection with the set of tangents to $\tilde{S}$ normal to L, and its intersection with $\tilde{S}$ itself. Since the area of this region is of the second order with respect to ϵ, the limit of the integral taken over it divided by ϵ is zero. The limit of the last integral is equal to the line integral

$$-\int_L F(x, y, z, \tilde{A}, \tilde{B}, \tilde{C})\omega(s)ds.$$

* Compare with Radon, loc. cit., p. 58, equation (15).

Since

$$\frac{\partial(\tilde{y}, \tilde{z})}{\partial(s, n)} = \tilde{A}, \qquad \frac{\partial(\tilde{z}, \tilde{x})}{\partial(s, n)} = \tilde{B}, \qquad \frac{\partial(\tilde{x}, \tilde{y})}{\partial(s, n)} = \tilde{C},$$

equation (2) may now be written

$$\delta J = -\epsilon \int_L E(x, y, z; A, B, C; \tilde{A}, \tilde{B}, \tilde{C})\omega(s)ds,$$

which does not vanish. Consequently there is no minimum. It follows that

$$E(x, y, z; A, B, C; \tilde{A}, \tilde{B}, \tilde{C}) = 0$$

at every point of L is a necessary condition for a minimum.

This condition is satisfied identically if S and $\tilde{S}$ have the same normal at each point of their intersection.

In the special case where x and y are the independent variables, and

$$J = \iint f(x, y, z, p, q)dxdy,$$

the function F becomes*

$$F(x, y, z, A, B, C) = f\left(x, y, z, \frac{-A}{C}, \frac{-B}{C}\right)C,$$

and $A = -p$, $B = -q$, $C = 1$. If these values are substituted in the function T,† it becomes

$$f_z - f_{px} - f_{qy} - f_{pz}p - f_{qz}q - f_{pp}\frac{\partial^2 z}{\partial x^2} - 2f_{pq}\frac{\partial^2 z}{\partial x \partial y} - f_{qq}\frac{\partial^2 z}{\partial y^2},$$

which is the left member of the Lagrange equation for the problem, and the Weierstrass E-function becomes

$$f(x, y, z, \tilde{p}, \tilde{q}) - f(x, y, z, p, q) - (\tilde{p} - p)f_p(x, y, z, p, q) - (\tilde{q} - q)f_q(x, y, z, p, q).$$

§ 3. Sufficient Conditions.

Sufficient conditions can be derived by a method similar to that used by Kneser‡ for the problem where the variations are unrestricted.

The first condition is that the given surface S_0 shall consist of an extremal surface, and a portion of $\tilde{S}$ the boundary of R, at every point of which the inequality (1) is satisfied, and the two parts of S_0 shall have the same normal at each point of their intersection. The second condition is that S_0 shall be embedded in a family of surfaces S_a, each consisting of an extremal surface and a part of S_0 or else of an extremal surface alone,

* Bolza, loc. cit., p. 665.

† Radon, loc. cit., p. 57.

‡ Kneser, Lehrbuch der Variationsrechnung, p. 300.

such that every point of R in the neighborhood of S_0 lies on one and only one extremal surface of the family, and such that any of the extremal surfaces and $\tilde{S}$ shall have the same normal at each point of their intersection. The last condition is that if A, B, C are proportional to the direction-cosines of the normal to the extremal surface and $\bar{A}$, $\bar{B}$, $\bar{C}$ proportional to the direction-cosines of any other line,

$$E(x, y, z; A, B, C; \bar{A}, \bar{B}, \bar{C}) > 0,$$

at every point in this neighborhood.

It will be proved that if these conditions are satisfied the surface S_0 gives a smaller value to the integral J than any other surface $\bar{S}$ of class D' which intersects S_0 along the boundary of the region over which the integral is taken, lies entirely in the neighborhood of S_0 defined above, and is not tangent to an infinite number of the surfaces S_a along parts of their intersections.

It can be assumed that no part of $\bar{S}$ is between S_0 and $\tilde{S}$ because the variations are unrestricted over the part of S_0 which does not coincide with $\tilde{S}$, and sufficient conditions for a minimum with respect to unrestricted variations are assumed. The surface $\bar{S}$ will intersect a certain number of the surfaces S_a in a set of closed curves L_a. The last surface S_a which it touches will be called S_{a_1}. The function $J(a)$ will be defined as the integral J taken over the part of $\bar{S}$ between L_0 and L_a and over the part of S_a bounded by L_a. Thus $J(0) = J(S_0)$ and $J(a_1) = J(\bar{S})$, and it must simply be proved that $J(a_1) - J(0)$ is positive.

If the equations of S_a are

$$S_a: \qquad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

the equations of $S_{a+\Delta a}$ can be put in the form

$$S_{a+\Delta a}: \qquad \begin{aligned} x &= x(u, v) + A\omega(u, v, \Delta a), \\ y &= y(u, v) + B\omega(u, v, \Delta a), \\ z &= z(u, v) + C\omega(u, v, \Delta a). \end{aligned}$$

The parameter a will be chosen in such a way that $a_1 > 0$ and the partial derivative $\omega_{\Delta a}(u, v, 0) > 0$ also. The equations of the part of $\bar{S}$ between L_a and $L_{a+\Delta a}$, excepting possibly in the neighborhood of points where the direction angles of L_a are discontinuous, can be put in the form

$$\Delta\bar{S}: \qquad x = \bar{x}(s, n), \qquad y = \bar{y}(s, n), \qquad z = \bar{z}(s, n),$$

where s is the length of arc measured along L_a, and n is the length of a curve normal to L_a. The value of n on $L_{a+\Delta a}$ will be called $N(s, \Delta a)$.

The equations of the projection of $\Delta\bar{S}$ on S_a will be expressed in terms of s and n', where n' is the length of the projection of an n-curve. If $N'(s, \Delta a)$ is the value of n' on the projection of $L_{a+\Delta a}$, and θ the angle between the normals to $\bar{S}$ and S_a, it is evident that the limit of N'/N is $\cos\theta$, if Δa approaches zero. The derivative of $J(a)$ is seen to be

$$\frac{dJ(a)}{da} = \lim_{\Delta a=0} \iint_{S_a} \frac{1}{\Delta a}(F(x + A\omega, y + B\omega, \cdots) - F(x, y, \cdots))dudv$$

$$+ \int_{L_a} (F(x, y, z, \bar{A}, \bar{B}, \bar{C}) - F(x, y, z, A, B, C)\cos\theta)N_{\Delta a}ds. \tag{3}$$

The first integral is equal to

$$\iint_{S_a} T(A^2 + B^2 + C^2)\omega_{\Delta a}dudv + \int_{L_a} \begin{vmatrix} F_A & F_B & F_C \\ A & B & C \\ \bar{x}_s(s, 0) & \bar{y}_s(s, 0) & \bar{z}_s(s, 0) \end{vmatrix} \omega_{\Delta a}ds.* \tag{4}$$

The existence of the partial derivative $N_{\Delta a}$ can be proved by finding its value in terms of $\omega_{\Delta a}$. At every point of $L_{a+\Delta a}$

$$x(s, N') + A\omega(s, N', \Delta a) = \bar{x}(s, N),$$
$$y(s, N') + B\omega(s, N', \Delta a) = \bar{y}(s, N),$$
$$z(s, N') + C\omega(s, N', \Delta a) = \bar{z}(s, N).$$

If these equations are differentiated with respect to Δa at $\Delta a = 0$, they become

$$A\omega_{\Delta a} = (\bar{x}_n - x_{n'}\cos\theta)N_{\Delta a},$$
$$B\omega_{\Delta a} = (\bar{y}_n - y_{n'}\cos\theta)N_{\Delta a},$$
$$C\omega_{\Delta a} = (\bar{z}_n - z_{n'}\cos\theta)N_{\Delta a}.$$

The coefficient of $N_{\Delta a}$ is different from zero in at least one of these equations unless $\bar{S}$ is tangent to S_a, and that case will be considered later. If these values are substituted for $A\omega_{\Delta a}$, $B\omega_{\Delta a}$ and $C\omega_{\Delta a}$ in the line integral in expression (4), it becomes

$$-\int_{L_a} [F_A(\bar{A} - A\cos\theta) + F_B(\bar{B} - B\cos\theta) + F_C(\bar{C} - C\cos\theta)]N_{\Delta a}ds.$$

Combining these results and making use of the identity

$$F(x, y, z, A, B, C) = AF_A + BF_B + CF_C,$$

* The sign of this line integral is different from the corresponding sign in equation (2), because the region considered is on the opposite side of the bounding curve.

we reduce equation (3) to

$$\frac{dJ(a)}{da} = \int\int_{S_a} T(A^2+B^2+C^2)\omega_{\Delta a}dudv + \int_{L_a} E(x,y,z;A,B,C;\bar{A},\bar{B},\bar{C})N_{\Delta a}ds.$$

The double integral cannot be negative and the line integral is positive. Consequently

$$\frac{dJ(a)}{da} > 0.$$

If S is tangent to S_a along a part of one and only one curve L_a, such as $L_{a'}$, and δ is an arbitrarily small positive number, the inequalities

$$J(a_1) - J(a' + \delta) > 0, \qquad J(a' - \delta) - J(0) > 0,$$

are satisfied, and both quantities increase as δ decreases. But $|J(a' + \delta) - J(a' - \delta)|$ can be made arbitrarily small by taking δ sufficiently small. It is evident then that

$$J(\bar{S}) - J(S_0) = J(a_1) - J(a) > 0.$$

A similar argument can be made if $\bar{S}$ is tangent to a finite number of the surfaces S_a. The case where it is tangent to an infinite number of them will not be considered here.

COLUMBIA UNIVERSITY.

A NEW PROOF OF SYLOW'S THEOREM.

By G. A. Miller.

The main part of Sylow's theorem relates to the fact that a necessary and sufficient condition that any group G of finite order g contain a subgroup whose order is p^α, p being a prime number, is that g is divisible by p^α. In particular, if p^α is the highest power of p which divides g, then G contains at least one subgroup of order p^α. This theorem was first proved by L. Sylow in the Mathematische Annalen, volume 5 (1872), page 584. Among the later proofs those by A. Capelli, E. Netto, and G. Frobenius are most noteworthy from the standpoint of originality and simplicity. The present writer has recently pointed out* that a simple proof of this theorem can also be obtained by modifying only slightly the method employed by A. Cauchy to prove the special but fundamental case that G contains a subgroup of order p whenever g is divisible by p.

Notwithstanding these various known proofs it seems desirable to direct attention to the following very elementary proof, especially since this proof follows almost directly from an elementary theorem which is of interest on its own account. This theorem may be stated as follows: *The number of the substitutions of degree p^β and of order p in the symmetric group of degree n is prime to p whenever p^β is the highest power of p which does not exceed n.*

For simplicity we shall first consider the case when $n = p^\beta$ in proving this theorem. It is evident that the subgroup of the symmetric group of degree p^β which is composed of all the substitutions of this symmetric group which are commutative with a substitution s of degree p^β and of order p is of order

$$p^{p^{\beta-1}} \cdot p^{\beta-1}!.$$

In fact, s is composed of $p^{\beta-1}$ cycles and these cycles are permuted according to the symmetric group of degree $p^{\beta-1}$ under the group which transforms s into itself.

All the factors of $p^\beta!$ which are divisible by p are evidently contained in the series

$$p, 2p, 3p, \cdots, p^{\beta-1} \cdot p.$$

Hence the order of this group involves the highest power of p which divides

* G. A. Miller, Bulletin of the American Mathematical Society, vol. 16 (1910), p. 511.

$p^\beta!$ and hence the number of conjugates of s under the symmetric group of degree p^β is prime to p.

Having proved the theorem in question when $n = p^\beta$ it is not difficult to prove it in general. In fact, the number of the substitutions which are similar to s and which are contained in the symmetric group of degree $p^\beta + k$ is evidently equal to

$$\frac{N(p^\beta + 1)(p^\beta + 2) \cdots (p^\beta + k)}{k!},$$

where N represents the number of these substitutions in the symmetric group of degree p^β. As $p^\beta + k$ and k are divisible by the same power of p whenever $p^\beta < p^\beta + k < p^{\beta+1}$, the theorem in question has been proved.

To prove Sylow's theorem by means of the theorem which has just been established let G be any group whose order g is divisible by p^α but not by $p^{\alpha+1}$, and represent G as a regular substitution group. Suppose that p^β is the highest power of p which is less than g, since the case when g is a power of p does not require consideration, and consider all the possible substitutions on the g letters of G which are of degree p^β and of order p. Since G is transitive, it cannot transform any of these substitutions into itself. It must therefore transform all of them into complete sets of conjugates under G such that each of these sets is composed of more than one substitution. As the total number of these substitutions is prime to p, at least one of these sets of conjugates involves a number m of substitutions, where m is prime to p.

Each of these m substitutions is transformed into itself by a subgroup of G whose order is g/m, where $m > 1$. Hence G contains a subgroup whose order is divisible by p^α. If this subgroup is of order p^α, our theorem is established. If it is not of this order, we have reduced our problem to that of a smaller group whose order is divisible by p^α. In case Sylow's theorem were not universally true it would clearly be possible to find a smallest group G for which it would not be satisfied. As the preceding considerations establish the fact that such a smallest group does not exist these considerations constitute a proof of Sylow's theorem.

The main elements which enter into the above proof, besides the given theorem, are the facts that every group of finite order can be represented as a regular substitution group, and that the total number of substitutions of a group which transform into itself any one of a system of more than one conjugates under this group constitute a subgroup whose index under this group is equal to the number of these conjugates. Both of these very elementary theorems were known long before Sylow's theorem was

first proved. As the proof of the italicized theorem is also based upon very elementary considerations the present proof is elementary as well as brief. It is not affected by the fact that G may be abelian or non-abelian.

For the sake of completeness it may be desirable to add that the italicized theorem mentioned above is a special case of the following: *A necessary and sufficient condition that the number of the substitutions of degree p^β and of order p in the symmetric group of degree n is divisible by p is that the coefficient of p^β is zero if n is written in the form $a_0 p^m + a_1 p^{m-1} + \cdots + a_m$, where the coefficients $a_0, a_1, \cdots, a_m$ are integers less than p and either positive or zero.* The proof of this theorem is similar to that of the special case given above. In the proof of Sylow's theorem we can evidently select for s any substitution of degree p^β and of order p such that the number of all its conjugates under the symmetric group of degree n is prime to p.

UNIVERSITY OF ILLINOIS.

NON-LINEAR INTEGRAL EQUATIONS OF THE VOLTERRA TYPE.

By Haig Galajikian.

Introduction.

An existence theorem concerning the non-linear integral equation of the Volterra type

$$u(x) = \phi(x) + \int_{x_0}^{x} f[x, t, u(t)]dt$$

was first given by T. Lalesco,* using the method of successive approximations after Picard. In 1911 this same equation was considered by the author,† independently of T. Lalesco's work. While the general plan of treatment was the same, the region of existence was made less restricted by the introduction of a curved boundary in place of the rectangular one used by T. Lalesco; also a new method for proving the uniqueness of the solution was introduced. The author also established existence theorems for the equation of more general type

$$u(x) = g\left\{x, \int_{x_0}^{x} f_1[x, t, u(t)]dt, \int_{x_0}^{x} f_2[x, t, u(t)]dt\right\},$$

the method used, that of successive approximations, applying to a system of n such equations with m arguments involving n unknown functions.

It is evident that the value of a theorem of this sort is considerably increased by the weakening of the conditions imposed on the functions involved and by increasing the size of intervals in which these conditions must hold. Hence, in the first part of this paper, the set of integral equations, mentioned above, is considered, and followed by an effort at an improvement of the existence theorems so far given by widening the interval in which the existence of solution has been established.

It may be that in certain cases it is not only required to know whether or not a solution exists, but whether or not the solution has derivatives; and this naturally comprises the question of continuity. Hence the purpose of the main body of this paper is two-fold. First, it is to show that

* Journal de Mathématiques, series 6, vol. 4 (1908), p. 165.

† Master's thesis, Cornell University, 1912; cf. Bulletin of American Mathematical Society, vol. 19 (1912), pp. 342–346.

the solutions given by the existence theorems will, under certain conditions, be continuous functions of both upper and lower limits of the integral arguments appearing in the equations. In other words, it is to show that under the conditions of the problems no discontinuous solution may exist. Secondly, it is to show the existence of first derivatives with respect to the upper and lower limits of integration, in other words with respect to the arguments of the unknown function. Here the methods used necessitate the introduction of such additional conditions as continuity and existence of certain of the partial derivatives of the functions involved in the equations treated.

The methods used throughout the paper are very simple and straightforward. The establishment of the first derivative of the solution with respect to the upper limit of integration is accomplished in a truly elementary way, void of all difficulties. However, the study of the first derivative of the solution, with respect to the lower limit of integration, necessitates the proof of the existence of a unique solution of an integral equation of a more general type than heretofore considered.

The third part of the paper is devoted to the study of this general non-linear integral equation, under restrictions very similar to those assumed in theorems given in the first part of the paper.

A short section is added at the end to show how the question of first derivatives may successfully be extended to derivatives of any order whatever.

As the forms and lengths of the equations treated are rather cumbersome in certain sections, a notation has been introduced, which, although not claimed to be perfect, does simplify the work to a considerable extent. In what follows an integral equation of the type

$$u(x, x_0) = g\left\{x, \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt, \cdots, \int_{x_0}^{x} f_m[x, t, u(t, x_0)]dt\right\}$$

will be denoted by the equation

$$u(x, x_0) = [x, \overset{*}{[}x, t \overset{x}{;}_{x_0} u]].$$

The absence of the star over the second bracket indicates that there is only one argument, in the function g, of the form

$$\int_{x_0}^{x} f[x, t, u(t, x_0)]dt.$$

The presence of the star shall mean that the process involved is absolutely the same for any finite number of arguments as for one, and the expression

represents the sum of a number of terms similar to the one written. Partial derivatives will be denoted by the symbols

$$_{(x)}[\cdots] \quad \text{and} \quad _{(x)}[\cdots{}_{(y)}[\cdots]].$$

In order to avoid all possibilities of confusion, we cite here an example from page 183 of the paper. There we have the expression

$$\begin{aligned}
&{}_{(x)}[x+\theta k, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u] + \theta\{\overset{*}{[}x+k, t \overset{x+k}{\underset{x_0}{;}} u] - \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]\}] \\
&\qquad + \frac{1}{k}\{\overset{*}{[}x+k, t \overset{x+k}{\underset{x_0}{;}} u] - \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]\}\,{}_{(x_m)}[x+\theta k, \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u] \\
&\qquad\qquad + \theta\{\overset{*}{[}x+k, t \overset{x+k}{\underset{x_0}{;}} u] - \overset{*}{[}x, t \overset{x}{\underset{x_0}{;}} u]\}]
\end{aligned}$$

which written out in full for two arguments v_1, v_2 would be

$$\begin{aligned}
&g_x\Big\{x+\theta k, \quad \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt + \theta\Big[\int_{x_0}^{x+k} f_1[x+k, t, u(t, x_0)]dt \\
&\quad - \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt\Big], \quad \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt \\
&\quad + \theta\Big[\int_{x_0}^{x+k} f_2[x+k, t, u(t, x_0)]dt - \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt\Big]\Big\} \\
&\quad + \frac{1}{k}\Big[\int_{x_0}^{x+k} f_1[x+k, t, u(t, x_0)]dt - \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt\Big] \cdot g_{v_1}\Big\{x+\theta k, \\
&\quad \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt + \theta\Big[\int_{x_0}^{x+k} f_1[x+k, t, u(t, x_0)]dt \\
&\quad - \int_{x_0}^{x} f_1[x, t, u(t, x_0)]dt\Big], \quad \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt \\
&\quad + \theta\Big[\int_{x_0}^{x+k} f_2[x+k, t, u(t, x_0)]dt - \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt\Big]\Big\} \\
&\quad + \frac{1}{k}\Big[\int_{x_0}^{x+k} f_2[x+k, t, u(t, x_0)]dt - \int_{x_0}^{x} f_2[x, t, u(t, x_0)]dt\Big] \\
&\qquad \times g_{v_2}\{\text{similar terms as in } g_{v_1}\}.
\end{aligned}$$

This example will show the advantage of the notation.

1. The System of Integral Equations

$$
\begin{aligned}
u_1(x, x_0) &= {}^1[x, \overset{*}{[}x, t\overset{x}{\underset{x_0}{;}} u_1, u_2, \cdots, u_n]], \\
u_2(x, x_0) &= {}^2[x, \overset{*}{[}x, t\overset{x}{\underset{x_0}{;}} u_1, u_2, \cdots, u_n]], \qquad (1) \\
&\cdots\cdots\cdots, \\
u_n(x, x_0) &= {}^n[x, \overset{*}{[}x, t\overset{x}{\underset{x_0}{;}} u_1, u_2, \cdots, u_n]].
\end{aligned}
$$

THEOREM. *Let* ${}^1[x, v_1, \cdots, v_m], \cdots, {}^n[x, v_1, \cdots, v_m]$ *be continuous and satisfy Lipschitz conditions for the arguments* $v_1, v_2, \cdots, v_m$, *namely*

$$
|{}^i[x, v_1 + \mu_1, v_2 + \mu_2 \cdots, v_m + \mu_m] - {}^i[x, v_1, v_2, \cdots, v_m]| \leqq A \sum_{k=1}^{m} |\mu_k| \qquad (i = 1, 2, \cdots n),
$$

in the region $|x - x_0| \leqq \alpha$, $|v_1| \leqq \beta$, $|v_2| \leqq \beta$, $\cdots$, $|v_m| \leqq \beta$. *Let* $\overset{*}{[}x, t; u_1, u_2, \cdots, u_n]$ *be continuous functions, and satisfy Lipschitz conditions for the arguments,* $u_1, u_2, \cdots, u_n$, *namely*

$$
|[x, t; u_1 + \nu_1, u_2 + \nu_2, \cdots, u_n + \nu_n] - [x, t; u_1, u_2, \cdots, u_n]| \leqq B \sum_{i=1}^{n} |\nu_i|
$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|u_i - {}^i[x, 0, 0, \cdots, 0]| \leqq \gamma$, $(i = 1, 2, \cdots, n)$; *furthermore let*

$$
|[x, t; u_1, u_2, \cdots, u_n]| \leqq M.
$$

Then if ρ *satisfies the conditions*

$$
\rho \leqq \alpha, \qquad \rho \leqq \frac{\beta}{M}, \qquad \rho \leqq \frac{\gamma}{mAM},
$$

there exists, in the interval $|x - x_0| \leqq \rho$, *one and only one system of continuous functions,*

$$
u_1(x, x_0), \qquad u_2(x, x_0), \qquad \cdots, \qquad u_n(x, x_0),
$$

satisfying the system of integral equations (1).

Define the functions

$$
\begin{aligned}
u_{i0}(x, x_0) &= {}^i[x, 0, 0, \cdots, 0] \qquad (i = 1, 2, \cdots, n] \\
u_{ik}(x, x_0) &= {}^i[x, \overset{*}{[}x, t\overset{x}{\underset{x_0}{;}} u_{1k-1}, u_{2k-1}, \cdots, u_{nk-1}]] \qquad (2) \\
&\qquad\qquad (i = 1, 2, \cdots, n), \ (k = 1, 2, \cdots).
\end{aligned}
$$

It is easily seen that at each stage of the approximation the conditions

$$|u_{ik}(x, x_0) - {}^{i}[x, 0, 0, \cdots, 0]| \leqq \gamma \quad (i = 1, 2, \cdots, n),$$

$$|[x, t \overset{x}{\underset{x_0}{;}} u_{1k}, u_{2k}, \cdots, u_{nk}]| \leqq \beta \qquad (k = 1, 2, \cdots),$$

are satisfied when $|x - x_0| \leqq \rho$, and hence the next following set of approximation will have a meaning. Evidently each $u_{ik}(x, x_0)$ is continuous for $|x - x_0| \leqq \rho$.

We shall prove that the sequences

$$u_{1k}(x, x_0), \qquad u_{2k}(x, x_0), \qquad \cdots, \qquad u_{nk}(x, x_0),$$

approach, uniformly, the limits

$$u_1(x, x_0), \qquad u_2(x, x_0), \qquad \cdots, \qquad u_n(x, x_0).$$

To this end we write,

$$u_{ik}(x, x_0) - u_{ik-1}(x, x_0) = U_{ik}(x, x_0),$$

so that

$$u_{ik}(x, x_0) = u_{i0}(x, x_0) + U_{i1}(x, x_0) + \cdots + U_{ik}(x, x_0),$$
$$(i = 1, 2, \cdots, n), \ (k = 1, 2, \cdots).$$

The Lipschitz conditions on ${}^{i}[x, v_1, v_2, \cdots, v_n]$ and $\overset{*}{[}x, t; u_1, u_2, \cdots, u_n]$ give us,

$$|U_{ik}(x, x_0)| \leqq \frac{mnAB}{n}\left|\int_{x_0}^{x}\sum_{i=1}^{n}|U_{ik-1}(t, x_0)|dt\right|$$

$$\leqq \frac{m^2n^2A^2B^2}{n}\left|\int_{x_0}^{x}\left|\int_{x_0}^{t}\sum_{i=1}^{n}|U_{ik-2}(t_1, x_0)|dt_1\right|dt\right|$$

$$\cdots \qquad \leqq \frac{m^{k-1}n^{k-1}A^{k-1}B^{k-1}}{n}\left|\int_{x_0}^{x}\left|\int_{x_0}^{t}\left|\int_{x_0}^{t_1}\right|\right|\right|$$

$$\cdots\left|\int_{x_0}^{t_{k-2}}\sum_{i=1}^{n}|U_{i1}(t_{k-1}, x_0)|dt_{k-1}\right|\cdots\left|dt_1\right|dt\Bigg|$$

$$\leqq \frac{m^kn^kA^kB^k}{nB}\left|\int_{x_0}^{x}\left|\int_{x_0}^{t}\right|\cdots\left|\int_{x_0}^{t_{k-2}}|M|t_{k-1} - x_0|dt_{k-1}\right|\cdots\left|dt_1\right|dt\right|$$

$$\leqq \left(\frac{M}{nB}\right)\cdot\left[\frac{(mnAB\rho)^k}{k!}\right] \quad \text{when} \quad |x - x_0| \leqq \rho.$$

Since the series of constants

$$\sum_{k=1}^{\infty}\left[\frac{(mnAB\rho)^k}{k!}\right]$$

converges, we see that the series

$$u_{i0}(x, x_0) + U_{i1}(x, x_0) + U_{i2}(x, x_0) + \cdots + U_{ik}(x, x_0) + \cdots$$

converge uniformly, $|x - x_0| \leqq \rho$; if we represent their values by

$$u_1(x, x_0), \quad u_2(x, x_0), \quad \cdots, \quad u_n(x, x_0),$$

we have

$$\lim_{k=\infty} u_{1k}(x, x_0) = u_1(x, x_0), \qquad \cdots, \qquad \lim_{k=\infty} u_{nk}(x, x_0) = u_n(x, x_0)$$

uniformly, $|x - x_0| \leqq \rho$. By reference to (2) we see that these limits satisfy the system (1).

That there is only one such set of continuous functions, satisfying the system (1), may be seen as follows. Suppose there were two sets of continuous functions $u_i(x, x_0)$ and $z_i(x, x_0)$ satisfying the set (1), and put

$$u_i(x, x_0) - z_i(x, x_0) = w_i(x, x_0).$$

If we now write the set of equations (1) for $u_i(x, x_0)$ and $z_i(x, x_0)$, and subtract, we see from conditions on the functions ${}^{i}[x, v_1, v_2, \cdots, v_m]$ and $\overset{*}{[}x, t; u_1, u_2, \cdots, u_n]$, that

$$|w_i(x, x_0)| \leqq mAB \left| \int_{x_0}^{x} |w_i(t, x_0)| dt \right|, \tag{3}$$

and if we write $|w_i(x, x_0)| \leqq W$, we have by successive applications of (3)

$$|w_i(x_1 x_0)| \leqq mnABW|x - x_0|, \qquad |w_i(x, x_0)| \leqq m^2 n^2 A^2 B^2 W \frac{(x - x_0)^2}{2!},$$

and in general

$$|w_i(x, x_0)| \leqq W \frac{[mnAB|x - x_0|]^k}{k!}.$$

The expression on the right has the limit zero as k becomes infinite, therefore

$$w_i(x, x_0) = 0, \qquad \text{or} \qquad u_i(x, x_0) = z_i(x, x_0).$$

2. **Removal of the condition**

$$|x - x_0| \leqq \rho \leqq \frac{\gamma}{mAM}.$$

In the preceeding article the existence of a set of continuous functions, satisfying the given set of integral equations, was shown in a region smaller than the one in which continuity and the Lipschitz conditions were imposed on the functions ${}^{i}[x, v_1, v_2, \cdots, v_m]$ and $\overset{*}{[}x, t; u_1, u_2, \cdots, u_n]$. More particularly, we had

$$|x - x_0| \leqq \rho; \qquad \rho \leqq \alpha, \qquad \rho \leqq \frac{\beta}{M}, \qquad \rho \leqq \frac{\gamma}{mAM}.$$

The removal of any one of these condition, on our interval, will undoubtedly add to the value of the theorem. The third of the conditions on ρ may be removed by a simple procedure, depending on the part of the existence theorem of the above section dealing with the uniqueness of the solution. As this condition for a system of equations of the type (1) is identically the same as for one equation of the form

$$u(x, x_0) = [x, [x, t \overset{x}{\underset{x_0}{;}} u]],$$

and as the argument employed is identically the same as for an equation

$$u(x, x_0) = [x, [x, t \overset{x}{\underset{x_0}{;}} u]],$$

where the third condition on ρ is

$$\rho \leqq \frac{\gamma}{AM},$$

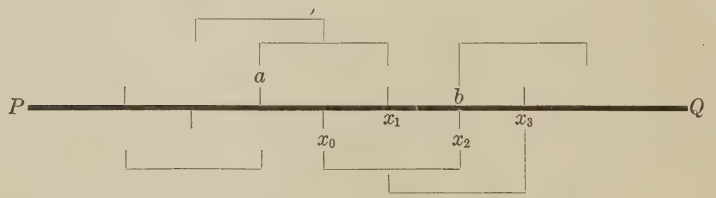
we shall treat this last equation here.

Let the line PQ represent the interval

$$|x - x_0| \leqq \min\left(\alpha, \frac{\beta}{M}\right),$$

and subdivide this line so that

$$x_\lambda - x_{\lambda-1} = \frac{\gamma}{AM}.$$



In any one of the subintervals $|x_\lambda - x_{\lambda-1}| \leqq \gamma/AM$ a solution of

$$u(x, x_0) = [x, [x, t \overset{x}{\underset{x_{\lambda-1}}{;}} u]]$$

exists and is unique. Let us call $u^0(x, x_0)$ the solution of

$$u^0(x, x_0) = [x, [x, t \overset{x}{\underset{x_0}{;}} u^0]],$$

for $|x - x_0| \leqq x_1 - x_0$; it exists and is unique. Now consider

$$(1) \qquad u^{(1)}(x, x_0) = [x, [x, \overset{x_1}{\underset{x_0}{t}} ; u^0] + [x, \overset{x}{\underset{x_1}{t}} ; u^{(1)}]] = [x, [x, t \overset{x}{\underset{x_1}{;}} u^{(1)}]]^{(1)}$$

an integral equation whose solution exists for $|x - x_1| \leqq x_1 - x_0$, and is unique, for

$$|[x, v + \mu]^{(1)} - [x, v]^{(1)}| \leqq A\mu,$$

and for a small enough subinterval

$$|[x, \overset{x}{\underset{x_0}{t}}; u^0] + v| \leqq \beta,$$

or

$$|v| \leqq \beta - M(x_1 - x_0),$$

and

$$|x - x_0| \leqq \alpha, \qquad |t - x_0| \leqq \alpha.$$

This may be seen from the approximations

$$u_0^{(1)}(x, x_0) = [x, 0]^{(1)},$$

$$u_\nu^{(1)}(x, x_0) = [x, [x, \overset{x}{\underset{x_1}{t}}; u_{\nu-1}^{(1)}]]^{(1)},$$

where

$$|[x, \overset{x}{\underset{x_1}{t}}; u_\nu^{(1)}]| \leqq \beta - M(x_1 - x_0),$$

or

$$M|x - x_1| \leqq \beta - M(x_1 - x_0),$$

which is evidently true for $x_1 \leqq x \leqq x_2$ and consequently also for $x_0 \leqq x \leqq x_1$, and we also have

$$|u_\nu^{(1)}(x, x_0) - [x, 0]^{(1)}| \leqq \gamma.$$

Suppose now $x_0 < x < x_1$, then

$$u^{(0)}(x, x_0) = [x, [x, \overset{x}{\underset{x_0}{t}}; u^{(0)}]] = [x, [x, \overset{x_1}{\underset{x_0}{t}}; u^{(0)}] + [x, \overset{x}{\underset{x_1}{t}}; u^{(0)}]] = [x, [x, \overset{x}{\underset{x_0}{t}}; u^{(0)}]]^{(1)}.$$

Therefore $u^0(x, x_0)$ is a solution of (1) for $x_0 < x < x_1$. But the solution of (1) is unique for $|x - x_1| \leqq (x_1 - x_0)$; it follows therefore that

$$u^{(0)}(x, x_0) \equiv u^{(1)}(x, x_0) \quad \text{for} \quad x_0 \leqq x \leqq x_1.$$

Therefore there is one and only one solution of our original integral equation in the extended interval (a, b). Continuing this process, we extend this solution to the interval PQ.

3. Continuity of the Solution.

THEOREM: *Given the integral equation*

(1) $$u(x, x_0) = [x, [x, \overset{x}{\underset{x_0}{t}}; u(t, x_0)]];$$

let $[x, v_1, v_2, \cdots, v_m]$ *be continuous and satisfy for all its arguments* the Lipschitz conditions*

$$|[x + \mu_0, v_1 + \mu_1, v_2 + \mu_2, \cdots, v_m + \mu_m] - [x, v_1, v_2, \cdots, v_m]| \leqq A\sum_{k=0}^{m}|\mu_k|$$

in the region $|x - x_0| \leqq \alpha$, $|v_1| \leqq \beta$, $|v_2| \leqq \beta$, $\cdots$, $|v_m| \leqq \beta$. *Let* $\overset{*}{[}x, t; u]$ *be continuous functions, and let them satisfy, for the arguments* x *and* u, *the Lipschitz conditions,*

$$|\overset{*}{[}x + \theta, t; u + \eta] - \overset{*}{[}x, t; u]| \leqq B\{|\theta| + |\eta|\},$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|u - [x, 0, \cdots, 0]| \leqq \gamma$. *Furthermore let*

$$|\overset{*}{[}x, t; u]| \leqq M.$$

Then if there is any solution whatever, in the interval $|x - x_0| \leqq \alpha$, *it is continuous in* x *and* x_0. In particular, then, the unique solution of the equation (1) in $|x - x_0| \leqq \rho$, $\rho \leqq \min(\alpha, \beta/M)$, is continuous in x and x_0 in this reduced interval. This also applies to the set of solutions of the system of equations of § 1.

We want to show that, given a positive number ϵ, however small, we can make the absolute value of the difference

$$u(x + k, x_0 + h) - u(x, x_0)$$

become and remain less than ϵ, by choosing $|k|$ and $|h|$ small enough.

Write

$$\begin{aligned}(2)\quad &|u(x + k, x_0 + h) - u(x, x_0)| \\ &\leqq \{|u(x + k, x_0 + h) - u(x, x_0 + h)| + |u(x, x_0 + h) - u(x, x_0)|\},\end{aligned}$$

the two parts of which may be made to become and remain less than $\epsilon/2$, respectively, for $|k|$ and $|h|$ small enough.

To this end we call

$$u_1(x, x_0) = [x, \overset{*}{[}x, \overset{z}{t}_{x_0+h}; u(t, x_0 + h)]],$$

$$u(x, x_0) = [x, \overset{*}{[}x, \overset{x}{t}_{x_0}; u(t, x_0)]],$$

and

$$z(x, x_0) = u_1(x, x_0) - u(x, x_0).$$

Breaking up (2) into its two parts we have,

* Notice here that an additional Lipschitz condition on the argument x is necessary whereas that was not at all necessary for the existence theorem in the form it appears in in this paper. See Bulletin of the American Mathematical Society, vol. 19 (1912), p. 345.

$$(a) \qquad |u(x+k, x_0+h) - u(x, x_0+h)| = |[x+k, \overset{*}{[}x+k, \overset{x+k}{\underset{x_0+h}{t}}; u_1]] - [x, \overset{*}{[}x, \overset{x}{\underset{x_0+h}{t}}; u_1]]|;$$

$$(b) \qquad |u(x, x_0+h) - u(x, x_0)| = |[x, \overset{*}{[}x, \overset{x}{\underset{x_0+h}{t}}; u_1]] - [x, \overset{*}{[}x, \overset{x}{\underset{x_0}{t}}; u]]|.$$

Taking (a) we have

$$|u(x+k, x_0+h) - u(x, x_0+h)| = |[x+k, \overset{*}{[}x+k, \overset{x}{\underset{x_0+h}{t}}; u_1] + \overset{*}{[}x+k, \overset{x+k}{\underset{x}{t}}; u_1]] - [x, \overset{*}{[}x, \overset{x}{\underset{x_0+h}{t}}; u_1]]|,$$

which, because of the Lipschitz condition on $[x, v_1, \cdots, v_m]$, is

$$\leqq A\{|k| + |\overset{*}{[}x+k, \overset{x}{\underset{x_0+h}{t}}; u_1] - \overset{*}{[}x, \overset{x}{\underset{x_0+h}{t}}; u_1]| + |\overset{*}{[}x+k, \overset{x+k}{\underset{x}{t}}; u_1]|\}$$

$$\leqq A\left\{|k| + \left|\int_{x_0+h}^{x} |\overset{*}{[}x+k, t; u_1] - \overset{*}{[}x, t; u_1]|dt\right| + \left|\int_{x}^{x+k} |\overset{*}{[}x+k, t; u_1]|dt\right|\right\},$$

which, because of the Lipschitz condition on $\overset{*}{[}x, t; u]$, is

$$\leqq A\left\{|k| + mB|k|\left|\int_{x_0+h}^{x} dt\right| + mM|k|\right\}$$

$$\leqq A\{|k| + mB|k(x - x_0 - h)| + mM|k|\} = A|k|\{1 + mB\alpha + mM\},$$

where

$$|x - x_0 - h| = \alpha.$$

Calling now $A\{1 + mB\alpha + mM\} = P$, where P is a finite constant, we get

$$|u(x+k, x_0+h) - u(x, x_0+h)| \leqq P|k|.$$

Treating (b) in the same manner, we have

$$|z(x, x_0)| = |[x, \overset{*}{[}x, \overset{x}{\underset{x_0+h}{t}}; u_1]] - [x, \overset{*}{[}x, \overset{x}{\underset{x_0}{t}}; u]]|$$

$$= |[x, \overset{*}{[}x, \overset{x}{\underset{x_0}{t}}; u_1] - \overset{*}{[}x, \overset{x_0+h}{\underset{x_0}{t}}; u_1]] - [x, \overset{*}{[}x, \overset{x}{\underset{x_0}{t}}; u]]|$$

$$\leqq A\left\{\left|\int_{x_0}^{x} |\overset{*}{[}x, t; u_1] - \overset{*}{[}x, t; u]|dt\right| + \left|\int_{x_0}^{x_0+h} |\overset{*}{[}x, t]; u|_1 dt\right|\right\}$$

$$\leqq A\left\{mB\left|\int_{x_0}^{x} |z|dt\right| + mM|h|\right\} = \lambda|w(x, x_0)| + \mu|h|,$$

where

$$\lambda = mAB, \qquad mAM = \mu, \qquad \int_{x_0}^{x} |z(t, x_0)|dt = w(x, x_0).$$

Thus we have the differential inequality

$$|w'(x, x_0)| \leqq \lambda|w(x, x_0)| + \mu|h|,$$

from which follows* that $w(x, x_0)$ satisfies the inequality

$$|w(x, x_0)| \leqq |w(x_0, x_0)|e^{\lambda|x-x_0|} + \frac{\mu|h|}{\lambda}(e^{\lambda|x-x_0|} - 1).$$

However,

$$w(x_0, x_0) = 0,$$

therefore

$$|w(x, x_0)| \leqq |h|\frac{\mu}{\lambda}(e^{\lambda|x-x_0|} - 1),$$

and finally from

$$w'(x, x_0) = |z(x, x_0)| \leqq \lambda|w(x, x_0)| + \mu|h|.$$

we have

$$|z(x, x_0)| \leqq \mu|h|e^{\lambda|x-x_0|}.$$

Combining this with the result of (a), we have

$$|u(x + k, x_0 + h) - u(x, x_0)| < \epsilon,$$

for $|k|$ and $|h|$ sufficiently small.

4. First Partial Derivatives of the Solution.

THEOREM: *Given the integral equation*

$$u(x, x_0) = [x, \overset{*}{[}x, t\ ;\ u]]; \tag{1}$$

let $[x, v_1, v_2, \cdots, v_m]$ *be continuous and satisfy Lipschitz conditions for the arguments* $v_1, v_2, \cdots, v_m$, *in the region* $|x - x_0| \leqq \alpha$, $|v_1| \leqq \beta$, $\cdots$, $|v_m| \leqq \beta$. *Let the first partial derivatives of* $[x, v_1, v_2, \cdots, v_m]$ *with respect to all its arguments exist and be continuous. Let* $\overset{*}{[}x, t; u]$ *be continuous functions and satisfy Lipschitz conditions for the argument* u *in the region* $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|u - [x, v_1, \cdots, v_m]| \leqq \gamma$. *Let the first partial derivatives of* $\overset{*}{[}x, t; u]$ *with respect to arguments* x *and* u *exist and be continuous in the same region. Furthermore let*

$$|\overset{*}{[}x, t; u]| \leqq M,$$

and

$$|[x, \overset{*}{[}x, t; u]]| \leqq N.$$

Then if $\rho \leqq \alpha$, $\rho \leqq \beta/M$, *the unique solution of* (1) *has first derivatives with respect to* x *and* x_0.†

* See for instance Bliss, Princeton Colloquium (1909), p. 88.

† It may be observed that the conditions, required for the existence of the first partial derivative with respect to x alone, are very much weaker than the ones stated above for both partial derivatives.

We want to show the existence of the following limits

(*a*) $$\operatorname*{Lim}_{k=0}\left[\frac{u(x+k,\,x_0)-u(x,\,x_0)}{k}\right],$$

and

(*b*) $$\operatorname*{Lim}_{h=0}\left[\frac{u(x,\,x_0+h)-u(x,\,x_0)}{h}\right].$$

Taking (*a*) first, we have

$$\frac{u(x+k,\,x_0)-u(x,\,x_0)}{k}=\frac{1}{k}\{[x+k,\,\overset{*}{[}x+k,\,t\,;\,u]]-[x,\,\overset{*}{[}x,\,t\,;\,u]]\}$$

$$=\frac{1}{k}\{[x+k,\,\overset{*}{[}x+k,\,t\,;\,u]+\overset{*}{[}x+k,\,t\,;\,u]]-[x,\,\overset{*}{[}x,\,t\,;\,u]]\},$$

which, by the laws of the mean for differentials and integrals, becomes

$$_{(x)}[x+\theta k,\,\overset{*}{[}x,\,t\,;\,u]+\theta\{\overset{*}{[}x+k,\,t\,;\,u]-\overset{*}{[}x,\,t\,;\,u]\}]$$

$$+\frac{1}{k}\{\overset{*}{[}x+k,\,t\,;\,u]-\overset{*}{[}x,\,t\,;\,u]\}\,{}_{(v_m)}[x+\theta k,\,\overset{*}{[}x,\,t\,;\,u]$$

$$+\theta\{\overset{*}{[}x+k,\,t\,;\,u]-\overset{*}{[}x,\,t\,;\,u]\}]=\,{}_{(x)}[x+\theta k,\,\overset{*}{[}x,\,t\,;\,u]$$

$$+\theta\{\overset{*}{[}x+k,\,t\,;\,u]-\overset{*}{[}x,\,t\,;\,u]+\overset{*}{[}x+k,\,t\,;\,u]\}$$

$$+\frac{1}{k}\{\overset{*}{[}x+k,\,t\,;\,u]-\overset{*}{[}x,\,t\,;\,u]+\overset{*}{[}x+k,\,t\,;\,u]\}\,{}_{(v_m)}[x+\theta k,\,\overset{*}{[}x,\,t\,;\,u]$$

$$+\theta\{\overset{*}{[}x+k,\,t\,;\,u]-\overset{*}{[}x,\,t\,;\,u]+\overset{*}{[}x+k,\,t\,;\,u]\}],$$

and

$$\frac{u(x+k,\,x_0)-u(x,\,x_0)}{k}=\,{}_{(x)}[x+\theta k,\,\overset{*}{[}x,\,t\,;\,u]+k\theta\{{}_{(x)}\overset{*}{[}x+\theta_1 k,\,t\,;\,u]$$

$$+\overset{*}{[}x+k,\,x+\theta_2 k,\,u(x+\theta_2 k,\,x_0)]\}]+\{{}_{(x)}\overset{*}{[}x+\theta_1 k,\,t\,;\,u]$$

$$+\overset{*}{[}x+k,\,x+\theta_2 k,\,u(x+\theta_2 k,\,x_0)]\}\cdot{}_{(v_m)}[x+\theta k,\,\overset{*}{[}x,\,t\,;\,u]$$

$$+k\theta\{{}_{(x)}\overset{*}{[}x+\theta_1 k,\,t\,;\,u]+\overset{*}{[}x+k,\,x+\theta_2 k,\,u(x+\theta_2 k,\,x_0)]\}],$$

where θ, θ_1, θ_2 are positive proper fractions. Passing to the limit, we have

$$\frac{\partial u(x,\,x_0)}{\partial x}=\,{}_{(x)}[x,\,\overset{*}{[}x,\,t\,;\,u]]+\{{}_{(x)}\overset{*}{[}x,\,t\,;\,u]+[x,\,x,\,u(x,\,x_0)]\}\cdot{}_{(v_m)}[x,\,\overset{*}{[}x,\,t\,;\,u]].$$

Again we have,

$$\frac{u(x, x_0 + h) - u(x, x_0)}{h} = \frac{1}{h}\{[x, \overset{*}{[}x, t \overset{x}{;}_{x_0+h} u(t, x_0 + h)]] - [x, \overset{*}{[}x, t \overset{x}{;}_{x_0} u]]\}$$

$$= \frac{1}{h}\{\overset{*}{[}x, t \overset{x}{;}_{x_0} u(t, x_0 + h)] - \overset{*}{[}x, t \overset{x}{;}_{x_0} u] - \overset{*}{[}x, t \overset{x_0+h}{;}_{x_0} u(t, x_0 + h)]\}$$

$$\times {}_{(v_m)}[x, \overset{*}{[}x, t \overset{x}{;}_{x_0} u] + \theta\{\overset{*}{[}x, t \overset{x}{;}_{x_0} u(t, x_0 + h)] - \overset{*}{[}x, t \overset{x}{;}_{x_0} u] - \overset{*}{[}x, t \overset{x_0+h}{;}_{x_0} u(t, x_0 + h)]\}]$$

$$= \frac{1}{h}\left\{\int_{x_0}^{x} {}_{(u)}\overset{*}{[}x, t\,; u(t, x_0 + \theta_1 h)](u(t, x_0 + h) - u(t, x_0))dt\right.$$

$$\left. - h\overset{*}{[}x, x_0 + \theta_2 h, u(x_0 + \theta_2 h, x_0 + \theta_1 h)]\right\} \cdot {}_{(v_m)}[x, \overset{*}{[}x, t \overset{x}{;}_{x_0} u]$$

$$+ \theta\left\{\int_{x_0}^{x} {}_{(u)}\overset{*}{[}x, t\,; u(t, x_0 + \theta_1 h)](u(t, x_0 + h) - u(t, x_0))dt\right.$$

$$\left. - h\overset{*}{[}x, x_0 + \theta_2 h, u(x_0 + \theta_2 h, x_0 + \theta_1 h)]\right\},$$

and taking the limit of both sides, we have

$$\operatorname*{Lim}_{h=0}\left[\frac{u(x, x_0 + h) - u(x, x_0)}{h}\right]$$

$$= \left\{\int_{x_0}^{x} {}_{(u)}[x, t\,; u] \cdot \operatorname*{Lim}_{h=0}\left[\frac{u(t, x_0 + h) - u(t, x_0)}{h}\right]dt\right.$$

$$\left. - \overset{*}{[}x, x_0, 0]\right\} {}_{(v_m)}[x, \overset{*}{[}x, t \overset{x}{;}_{x_0} u]].$$

Assuming that the limit, appearing on either side, exists, and denoting it by

$$z(x, x_0) = \frac{\partial u(x, x_0)}{\partial x_0},$$

we have

$$u(x, x_0) = -\int_{x_0}^{x} z(x, \xi)d\xi,$$

and substituting above, we have

$$(1)' \quad z(x, x_0) = \left\{ {}_{(u)}\overset{*}{\left[}\left(x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z(t, \xi)d\xi\right)z(t, x_0)\right]\right.$$

$$\left. - \overset{*}{[}x, x_0, 0]\right\} \cdot {}_{(v_m)}\left[x, \left[x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z(t, \xi)d\xi\right]\right],$$

which is an integral equation of a type not considered heretofore.

Now if the assumption about the limit of

$$\frac{u(x, x_0 + h) - u(x, x_0)}{h},$$

as h becomes indefinitely small, is true, a formal integration of $(1)^1$, with respect to x_0, ought to lead us back to equation (1). If this does happen, then to make the theorem complete, we must prove the existence of a unique solution of $(1)^1$ under the conditions of the problem. But $(1)^1$, as it stands, is the result of a formal differentiation of (1), with respect to x_0, and we therefore proceed to the existence theorem of a unique solution of $(1)^1$ which we write in the more general form:

$$z(x, x_0) = \left\{\left[\left(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z(t, \xi)d\xi\right)z\right] - {}^{*}[x, x_0, 0]\right\} \cdot {}^{i}\left[x, \left[\left(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z(t, \xi)d\xi\right)\right]\right] \quad (i = 1, 2, \cdots, m). \tag{2}$$

Define the functions

$$z_0(x, x_0) = - {}^{*}[x, x_0, 0]^{(i)}[x, 0, 0],$$

$$z_1(x, x_0) = \left\{\left[\left(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{x} z_0(t, \xi)d\xi\right)z_0\right] - {}^{*}[x, x_0, 0]\right\} \times {}^{i}\left[x, \left[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right]\right],$$

.,

$$z_n(x, x_0) = \left\{\left[\left(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_{n-1}(t, \xi)d\xi\right)z_{n-1}\right] - {}^{*}[x, x_0, 0]\right\} \times {}^{i}\left[x, \left[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_{n-1}(t, \xi)d\xi\right]\right],$$

.,

and such that

$$\int_{x_0}^{x} z_0(x, \xi)d\xi = \int_{x_0}^{x} - {}^{*}[x, \xi, 0]^{(i)}[x, 0, 0]d\xi = [x, 0, 0] \quad \text{of (1).}$$

It is evident that at each stage of the approximation the conditions

$$\left| -\int_{x_0}^{x} z_n(x, \xi)d\xi - [x, 0, 0] \right| \leqq \gamma \qquad (n = 1, 2, \cdots),$$

and

$$\left|\left[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_n(t, \xi)d\xi\right]\right| \leqq \beta$$

are satisfied for $|x - x_0| \leqq \rho$.

We need to show the uniform convergence, toward a definite limit, of the above defined sequence of approximate expressions—a limit that will satisfy (2), as is evident from the mode of the approximations.

To this end we write

$$z_n(x, x_0) - z_{n-1}(x, x_0) = Z_n(x, x_0) \qquad (n = 1, 2, \cdots),$$

so that

$$z_n(x, x_0) = z_0(x, x_0) + \sum_{i=1}^{n} Z_i(x, x_0) \qquad (n = 1, 2, \cdots).$$

Now if we let N be the larger of the maximum absolute values of ${}^{(i)}[x, v_1, \cdots, v_m]$; M, the larger of the maximum absolute values of

$$\left[\left(x, t ; -\int_{x_0}^{t} z_n(t, \xi)d\xi\right)z_n\right],$$

we have

$$|Z_1(x, x_0)| = |z_1(x, x_0) - z_0(x, x_0)| \leqq \left|\left\{\left[\left(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right)z_0\right]\right.\right.$$

$$\left.\left. - [x, x_0, 0]\right\}{}^{i}\left[x, \left[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right]\right] - \{ - [x, x_0, 0)]^{(i)}[x, 0, 0]\}\right|$$

$$\leqq \left|\left\{\left[\left(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right)z_0\right] - [x, x_0, 0]\right\}{}^{i}\left[x, \left[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right]\right]\right.$$

$$\left. - \{ - [x, x_0, 0]\}{}^{i}\left[x, \left[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right]\right]\right|$$

$$+ \left|\{ - [x, x_0, 0]\}{}^{i}\left[x, \left[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right]\right] - \{ - [x, x_0, 0]^{(i)}[x, 0, 0]\right|$$

$$\leqq \left\{\left|\left[\left(x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right)z_0\right]\right|\left|{}^{i}\left[x, \left[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right]\right]\right|\right.$$

$$\left. + |\{ - [x, x_0, 0]|\left|{}^{i}\left[x, \left[x, t \overset{x}{\underset{x_0}{;}} -\int_{x_0}^{t} z_0(t, \xi)d\xi\right]\right] - {}^{(i)}[x, 0, 0]\right|\right.$$

$$\leqq mN^2M|x - x_0| + m^2AM^2(x - x_0) = m[N^2M + mAM^2]|x - x_0|,$$

which we call ${}_{P_1}|x - x_0|$.

Now

$$|Z_2(x, x_0)| \leqq \Big| \Big\{ \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big) z_1 \Big]$$

$$- \overset{*}{[}x, x_0, 0] \Big\} \overset{i}{\Big[} x, \Big[x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big] \Big] - \Big\{ \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_0(t, \xi) d\xi \Big) z_0 \Big]$$

$$- \overset{*}{[}x, x_0, 0] \Big\} \overset{i}{\Big[} x, \Big[x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big] \Big] \Big| + \Big| \Big\{ \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_0(t, \xi) d\xi \Big) z_0 \Big]$$

$$- \overset{*}{[}x, x_0, 0] \Big\} \overset{i}{\Big[} x, \Big[x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big] \Big] - \Big\{ \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_0(t, \xi) d\xi \Big) z_0 \Big]$$

$$- \overset{*}{[}x, x_0, 0] \Big\} \overset{i}{\Big[} x, \Big[x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_0(t, \xi) d\xi \Big] \Big] \Big| \leqq \Big| \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big) z_1 \Big]$$

$$- \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_0(t, \xi) d\xi \Big) z_0 \Big] \Big| \Big| \overset{i}{\Big[} x, \Big[x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big] \Big] \Big|$$

$$+ \Big| \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_0(t, \xi) d\xi \Big) z_0 \Big] - \overset{*}{[}x, x_0, 0] \Big| \Big| \overset{i}{\Big[} x, \Big[x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big] \Big]$$

$$- \overset{i}{\Big[} x, \Big[x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_0(t, \xi) d\xi \Big] \Big] \Big| \leqq N \Big| \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big) z_1 \Big]$$

$$- \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big) z_0 \Big] \Big| + \Big| \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big) z_0 \Big]$$

$$- \Big[\overset{*}{\Big(} x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_0(t, \xi) d\xi \Big) z_0 \Big] \Big| + m\{NM|x - x_0| + M\}A$$

$$\times \Big| \Big[x, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_1(t, \xi) d\xi \Big] - \Big[\overset{*}{x}, t \overset{x}{;}_{x_0} - \int_{x_0}^{t} z_0(t, \xi) d\xi \Big] \Big|$$

$$\leqq mNMP_1 \frac{|x - x_0|^2}{2!} + \{mN^2B + m^2(NM|x - x_0| + M)AB\} \frac{P_1|x - x_0|^3}{3!}.$$

or

$$|Z_2(x, x_0)| \leqq mMNP_1 \frac{|x - x_0|^2}{2} + \left\{ \frac{P_1B}{M} + m^2MNAB|x - x_0| \right\} \frac{P_1|x - x_0|^3}{3}.$$

Similarly it may be shown that

$$|Z_3(x, x_0)| \leqq mMNP_2 \frac{|x - x_0|}{3} + \left\{ \frac{P_1B}{M} + m^2MNAB|x - x_0| \right\} P_2 \frac{|x - x_0|^2}{4},$$

and in general

$$|Z_n(x, x_0)| \leqq mMNP_{n-1}\frac{|x-x_0|}{n} + \left\{\frac{P_1B}{M} + m^2MNAB|x-x_0|\right\}\frac{P_{n-1}|x-x_0|^2}{n+1},$$

where P_n is the quantity such that

$$|Z_n(x, x_0)| < P_n.$$

Consequently, for

$$|x - x_0| \leqq \rho, \qquad \frac{P_n}{P_{n-1}} \leqq mMN\frac{\rho}{n} + \left(\frac{P_1B}{M} + m^2MNAB\rho\right)\frac{\rho^2}{n+1},$$

so that

$$\lim_{n=\infty}\frac{P_n}{P_{n-1}} = 0;$$

therefore,

$$\lim_{n=\infty} z_n(x, x_0) = z(x, x_0)$$

uniformly, and this limit $z(x, x_0)$ is a solution of (2).

That this solution is unique may be seen as follows. Suppose if possible there were two continuous solutions $z(x, x_0)$ and $z'(x, x_0)$ and call

$$w(x, x_0) = z(x, x_0) - z'(x, x_0).$$

It is evident that $w(x, x_0)$ is limited. If we write

$$|w(x, x_0)| \leqq W,$$

we can show, by methods similar to those used in the main body of the existence theorem, that

$$(3) \qquad \begin{aligned} |w(x, x_0)| \leqq{}& mNM\left|\int_{x_0}^{x}|z(t, x_0) - z'(t, x_0)|dt\right| \\ &+\left[\frac{PB}{M} + Q|x - x_0|\right]\left|\int_{x_0}^{x}\left|\int_{x_0}^{t}|z(t, \xi) - z'(t, \xi)|d\xi\right|dt\right|, \end{aligned}$$

where

$$P = mMN^2 + m^2M^2A \qquad \text{and} \qquad Q = m^2MNAB.$$

From (3) we have

$$|w(x, x_0)| \leqq \left\{mMN + \frac{PB}{M}|x - x_0| + Q|x - x_0|^2\right\}W|x - x_0|.$$

Substituting this last result in (3), we get

$$|w(x, x_0)| \leqq mMN \frac{|x - x_0|^2}{2}\left[mMNW + \frac{PBW}{M}|x - x_0| + QW|x - x_0|^2 \right]$$

$$+\left[\frac{PB}{M} + Q|x - x_0|\right]\left[mMNW + \frac{PBW}{M}|x - x_0| + QW|x - x_0|^2\right]\frac{|x-x_0|^3}{3!}$$

$$= \left[mMNW + \frac{PBW}{M}|x - x_0| + QW|x - x_0|^2 \right]\left[mNM \frac{|x - x_0|^2}{2!} \right.$$

$$\left. + \left\{ \frac{PB}{M} + Q|x - x_0| \right\} \right] \frac{|x - x_0|^3}{3!} \leqq \left[mMN + \frac{PB}{M}|x - x_0| \right.$$

$$\left. + Q|x - x_0|^2 \right]^2 \frac{W|x - x_0|^2}{2!}.$$

Repeating this process, we have

$$|w(x, x_0)| \leqq \left[mMN + \frac{PB}{M}|x - x_0| + Q|x - x_0|^2 \right]^3 \frac{W|x - x_0|^3}{3!},$$

and in general,

$$|w(x, x_0)| \leqq \left[mMN + \frac{PB}{M}|x - x_0| + Q|x - x_0|^2 \right]^n \frac{W|x - x_0|^n}{n!}.$$

But the expression on the right has the limit zero as n becomes indefinitely large. Therefore there is one and only one solution of (2) and hence of (1)[1] under the conditions of the problem.

5. The Integral Equation

$$u(x, x_0) = g\left\{ x, \int_{x_0}^{x} f_1\left[x, t, \int_{x_0}^{t} u(t, \xi)d\xi \right] u(t, x_0)dt, \right.$$

(1)

$$\left. \cdots, \quad \int_{x_0}^{x} f_m\left[x, t, \int_{x_0}^{t} u(t, \xi)d\xi \right] u(t, x_0)dt \right\}.$$

THEOREM: *Let* $g\{x, v_1, v_2, \cdots, v_m\}$ *be continuous and satisfy, for its arguments* $v_1, v_2, \cdots, v_m$, *the Lipschitz condition*

$$|g\{x, v_1 + \mu_1, v_2 + \mu_2, \cdots, v_m + \mu_m\} - g\{x, v_1, v_2, \cdots, v_m\}| \leqq A \sum_{k=1}^{m} |\mu_k|$$

in the region $|x - x_0| \leqq \alpha$, $|v_1| \leqq \beta$, $|v_2| \leqq \beta$, $\cdots$, $|v_m| \leqq \beta$; *furthermore let*

$$|g\{x, v_1, v_2, \cdots, v_m\}| \leqq N.$$

Let $f_1[x, t, \theta]$, $f_2[x, t, \theta]$, $\cdots$, $f_m[x, t, \theta]$ *be continuous and satisfy, for the argument* θ, *the Lipschitz conditions*

$$|f_k[x, t, \theta + \nu] - f_k[x, t, \theta]| \leqq B|\nu| \quad (k = 1, 2, \cdots, m),$$

in the region $|x - x_0| \leqq \alpha$, $|t - x_0| \leqq \alpha$, $|u - g\{x, 0, 0, \cdots, 0\}| \leqq \gamma$; *furthermore let*

$$|f_k[x, t, \theta]| \leqq M \qquad (k = 1, 2, \cdots, m).$$

Then if ρ *satisfies the conditions*

$$\rho \leqq \alpha, \qquad \rho \leqq \frac{\beta}{MN},^*$$

there exists, in $|x - x_0| \leqq \rho$, *one and only one continuous solution of the equation* (1).

Let us write (1) in the form

$$(1)^1 \qquad u(x, x_0) = [x, \overset{*}{[}(x, t \overset{x}{;}_{x_0} D)u]].$$

Define

$$u_0(x, x_0) = [x, 0, 0, \cdots, 0],$$

$$u_1(x, x_0) = [x, \overset{*}{[}(x, t \overset{x}{;}_{x_0} D_0)u_0]],$$

$$\cdots \cdots \cdots \cdots \cdots \cdots$$

$$u_n(x, x_0) = [x, \overset{*}{[}(x, t \overset{x}{;}_{x_0} D_{n-1})u_{n-1}]],$$

$$\cdots \cdots \cdots \cdots \cdots \cdots,$$

where the conditions

$$|u_n(x, x_0) - [x, 0, 0, \cdots, 0]| \leqq \gamma$$

and

$$|\overset{*}{[}(x, t \overset{x}{;}_{x_0} D_n)u_n]| \leqq \beta \qquad (n = 1, 2, \cdots)$$

are satisfied as may easily be seen by methods similar to those used in previous sections. Thus each successive approximation is justified.

To show the uniform convergence of the sequence of functions, defined above, to a definite limit which is, as seen by the forms of the approximations, a solution of the equation $(1)^1$ or (1), we write,

$$u_n(x, x_0) - u_{n-1}(x, x_0) = U_n(x, x_0) \quad (n = 1, 2, \cdots),$$

so that

$$u_n(x, x_0) = u_0(x, x_0) + U_1(x, x_0) + \cdots + U_n(x, x_0) \quad (n = 1, 2, \cdots).$$

We have

* Notice that the condition $\rho \leqq \gamma/mAMN$ is not introduced here, since it may be removed as in section 2.

$$|U_1| \leqq mANM|x - x_0|$$

$$|U_2| \leqq |[x, \overset{*}{[}(x, t \overset{x}{;}_{x_0} D_1)u_1]] - [x, \overset{*}{[}(x, t \overset{x}{;}_{x_0} D_0)u_0]]|$$

$$\leqq A\{|\overset{*}{[}(x, t \overset{x}{;}_{x_0} D_1)u_1] - \overset{*}{[}(x, t \overset{x}{;}_{x_0} D_1)u_0]|$$

$$+ |\overset{*}{[}(x, t \overset{x}{;}_{x_0} D_1)u_0] - \overset{*}{[}(x, t \overset{x}{;}_{x_0} D_0)D_0]|\}$$

$$\leqq A\left\{\left|\int_{x_0}^{x} |[u_1(t, x_0) - u_0(t, x_0)]\overset{*}{[}x, t; D_1]|dt\right|\right.$$

$$\left. + \left|\int_{x_0}^{x} |\overset{*}{[}x, t; D_1] - \overset{*}{[}x, t; D_0]u_0|dt\right|\right\}$$

$$\leqq m^2A^2MN\frac{|x - x_0|^2}{2!}[M + BN|x - x_0|],$$

$$|U_3| \leqq m^3A^3MN\frac{|x - x_0|^3}{3!}[M + BN|x - x_0|]^2,$$

.

$$|U_n| \leqq m^nA^nMN\frac{|x - x_0|^n}{n!}[M + BN|x - x_0|]^{n-1},$$

.

and the series

$$u_0 + mAMN\rho + m^2A^2MN\frac{\rho^2}{2!}[M + BN\rho] + \cdots$$

$$+ m^nA^nMN\frac{\rho^n}{n!}[M + BN\rho]^{n-1} + \cdots$$

being obviously convergent, we have

$$\operatorname*{Lim}_{n=\infty} u_n(x, x_0) = u(x, x_0)$$

uniformly for $|x - x_0| \leqq \rho$. This limit evidently satisfies (1)[1] and (1).

That there is only one such solution may be seen as follows. Suppose there were two, $u(x, x_0)$ and $v(x, x_0)$, and call the difference

$$w(x, x_0) = u(x, x_0) - z(x, x_0).$$

Then

$$|w(x, x_0)| \leqq W,$$

and we have

$$|w(x, x_0)| \leqq A\{|\overset{*}{[}(x, t \overset{x}{;}_{x_0} D_1)u] - \overset{*}{[}(x, t \overset{x}{;}_{x_0} D_2)v]|\}$$

$$\leqq A\{|\overset{*}{[}(x, t \overset{x}{;}_{x_0} D_1)u] - \overset{*}{[}(x, t \overset{x}{;}_{x_0} D_1)v]|$$

$$+ |\overset{*}{[}(x, t \overset{x}{;}_{x_0} D_1)v] - \overset{*}{[}(x, t \overset{x}{;}_{x_0} D_2)v]|\},$$

where

$$D_1 = \int_{x_0}^{t} u(t, \xi)d\xi \qquad \text{and} \qquad D_2 = \int_{x_0}^{t} v(t, \xi)d\xi.$$

Finally we have

$$|w(x, x_0)| \leqq A\left\{mM\left|\int_{x_0}^{x} |w(t, x_0)|dt\right| + mBN\left|\int_{x_0}^{x}\left|\int_{x_0}^{t} |w(t, \xi)|d\xi\right|dt\right|\right\}$$

(a)

$$\leqq A\left\{mMW|x - x_0| + mNBW\frac{|x - x_0|^2}{2!}\right\};$$

substituting this last result in (a) we have

$$|w(x, x_0)| \leqq m^2A^2M^2W\frac{|x - x_0|^2}{2!} + 2m^2A^2MBNW\frac{|x - x_0|^3}{3!}$$

$$+ m^2A^2MNBW\frac{|x - x_0|^4}{4!} \leqq m^2A^2W\frac{|x - x_0|^2}{2!}[M + NB|x - x_0|]^2.$$

Repeating this process, we get

$$|w(x, x_0)| \leqq m^nA^nW\frac{|x - x_0|^n}{n!}[M + NB|x - x_0|]^n,$$

which, as n becomes indefinitely large, approaches zero as a limit, therefore $w(x, x_0) = 0$. Consequently

$$u(x, x_0) = z(x, x_0).$$

PRINCETON UNIVERSITY

ON SOME GENERAL THEOREMS CONCERNING ORDINARY CLOSED CURVES.*

By Arnold Emch.

In an article which appeared in the Rendiconti del Circolo Matematico di Palermo† it is proved analytically that every closed analytic curve, without any other singularities than a finite number of inflexions, has an even number of inflexions and admits of an even number of tangents parallel to any given direction. The proofs of these theorems are based upon the coperiodicity of the parametric representations of such curves.

The method, these results and others, may be extended to any closed analytic curve without rectilinear segments and with a finite number of singularities of an algebraic character.

1. Theorems on Periodic Functions. To prove the more general theorems I shall make use of the following simple propositions on functions of a real variable:

Theorem I. *Given a real periodic function $u = F(t)$ of the real variable t with the primitive period interval $0 \leqq t < w$, which is uniform and continuous for all values of t in the interval, except for a finite number ν of values $t_1, t_2, t_3, \cdots, t_k, \cdots, t_\nu$, for which u becomes infinite in such a manner that when ϵ is an arbitrarily small positive real number,*

$$\lim_{\epsilon \doteq 0} F(t_k - \epsilon) \qquad \text{and} \qquad \lim_{\epsilon \doteq 0} F(t_k + \epsilon)$$

become one $+\infty$, the other $-\infty$. Assume furthermore that u within the period-interval vanishes for a finite number μ of values of t. Under these conditions $\nu + \mu$ is an even number. When $\nu = 0$, i. e. when u does not become infinite within the period-interval, then μ, the number of roots, when finite, is even.

There is no loss in generality in the assumption of single zeros for u, and $t_k \neq 0$, $k = 1, 2, 3, \cdots, \nu$.

Theorem 2. *Assume again a function $G(t)$ as defined above, but without infinities within the period-interval; moreover that the derivatives of $G(t)$ are functions of the same type as $G(t)$. Then every derivative of $G(t)$ is coperiodic with $G(t)$ and has at least two, and generally an even number of roots.*

* Read before the American Mathematical Society at Chicago, Dec. 28, 1914.

† Vol. 38 (1914), pp. 180–184.

2. Definition of a Closed Curve. Any closed curve may be represented parametrically in the form

$$x = \phi(t), \quad y = \psi(t), \tag{1}$$

where $\phi(t)$ and $\psi(t)$ are uniform, continuous real functions for all values of the real parameter t, and with the same period w.* It is moreover assumed that the curve has a continuous and uniform tangent for a continuous and uniform change of the parameter. The system of Cartesian coördinates (x, y) may always be chosen in such a manner, that the tangent at no singular point becomes parallel to either coördinate axis, and that $\phi(t)$ and $\psi(t)$ and all their derivatives are functions with the properties stated in Theorem 2. In the definition of the closed curve (1) we exclude rectilinear segments, and admit only arcs with singularities of an algebraic type, which shall be finite in number. Such a curve is called an *ordinary closed curve.*

3. Tangents From a Point to an Ordinary Closed Curve. Take any point (x_1, y_1) in the plane of the curve as defined, and not on the curve. The equation of the tangent at the point (x, y) of the curve is

$$\eta - y = \frac{dy}{dx}(\xi - x),$$

ξ and η denoting current coördinates. The condition that the tangent shall pass through (x_1, y_1) is, by substituting for x and y their parametric expressions and reducing,

$$[\phi(t) - x_1]\frac{d\psi}{dt} - [\psi(t) - y_1]\frac{d\phi}{dt} = 0. \tag{2}$$

The left-hand member of this equation is evidently a function of the type $G(t)$ as defined in Theorem 2, and vanishes therefore for no value or an even number of values of t within the period-interval. It is known that the values of t for which $\phi'(t)$ and $\psi'(t)$ vanish simultaneously define the cusps of the curve. These are therefore contained among the roots of (2). All the other roots determine proper tangents. A tangent at a point determined by a multiple root must be counted as often as the degree of multiplicity indicates. Thus, counting multiplicities properly, we have

THEOREM 3. *The number of tangents from a fixed point to an ordinary closed curve and its cusps is even, including, by definition, zero as an even number.*

When there are no cusps, then the number of tangents is even. When (x_1, y_1) is on the curve, the tangent at this point must be counted twice;

* See W. F. Osgood, Lehrbuch der Funktionentheorie, vol. I, 2d edition, pp. 140–150.

likewise, when the tangent through (x_1, y_1) is an inflectional tangent. A proper double-tangent through (x_1, y_1) is determined by two distinct values t_1 and t_2, and must, of course, be counted twice.

4. Inflexions and Cusps of Ordinary Closed Curves. Consider the function of t, formed from the parametric expressions of the curve (1),

$$W = \frac{u}{v},$$

where

$$u = \frac{d^2y}{dt^2} - \frac{dy}{dx} \cdot \frac{d^2x}{dt^2}, \qquad v = \left(\frac{dx}{dt}\right)^2.$$

The curve has inflexions for those values of t for which W vanishes. It is evident that u, v, W are all coperiodic with $\phi(t)$ and $\psi(t)$. As the latter and all their derivatives remain finite for all values of t, u can become infinite only when dy/dx becomes infinite. This will be the case when $dx/dt = 0$, $dy/dt \neq 0$, simultaneously. Hence, for such values $t = t_\tau$, for which this is the case, $|u| = +\infty$, and the curve has tangents parallel to the y-axis. In the neighborhood of such a point, as t increases continuously from $t_\tau - \epsilon$ to $t_\tau + \epsilon$, dy/dx changes abruptly from $\pm\infty$ to $\mp\infty$, while d^2x/dt^2 and d^2y/dt^2 remain finite. For all other values of t, different from t_τ, u is finite and continuous. u is therefore exactly a function as defined under Theorem 1, and admits therefore of an even number

$$2k = \mu + \nu$$

of zeros and infinities within the period interval, as specified. Points where we have $W = \infty$, and $dy/dx \neq \infty$ are defined as cusps. These are distinguished as cusps of the first kind, in the neighborhood of which d^2y/dx^2 changes signs; and cusps of the second kind, in the neighborhood of which d^2y/dx^2 does not change sign. As v is always positive, u, which remains finite, must change signs in case of a cusp of the first kind. In other words the cusps of the first kind are determined by a part c_1 of the number of roots of $u = 0$. Thus, the condition for a cusp of the first kind may be briefly stated by $u = 0$, $dx/dt = 0$. The remaining number i of roots of $u = 0$ determines the number of inflexions ($u = 0$, $dx/dt \neq 0$). The number of roots of u is therefore made up of c_1 and i, or $\mu = i + c_1$, so that according to Theorem 1

$$2k = i + c_1 + \nu. \tag{3}$$

Hence

THEOREM 4. *The number of tangents parallel to a given direction, inflexions and cusps of the first kind of an ordinary closed curve is even.*

When such tangents occur at cusps of the second kind, they must be included in this number.

In case of a cusp of the second kind, $u \neq 0$ and finite, and $dx/dt = 0$. The number c_2 of cusps of the second kind is contained in the number $2n$ of roots of $v = 0$, or $dx/dt = 0$, which, since $\phi'(t)$ is a periodic function as described under Theorem 2, is even. The number $2n$ is composed of the number ν of tangents parallel to the y-axis, the numbers c_1 and c_2 of cusps of the first and second kind; i. e.,

$$2n = c_1 + c_2 + \nu. \tag{4}$$

From (4) and (3) follows

$$i + c_2 = 2(n + k - \nu - c_1), \tag{5}$$

which is an even number. Hence

THEOREM 5. *An ordinary closed curve has an even number of inflexions and cusps of the second kind.*

Subtracting (4) from (3), we get

$$i - c_2 = 2(k - n), \tag{6}$$

an even number. Hence

THEOREM 6. *The difference between the number of inflexions and the number of cusps of the second kind of an ordinary closed curve is even.*

5. Algebraic Curves. When the closed curve is a complete irreducible algebraic curve, then its order is even and the theorem on inflexions follows immediately from Plücker's formulas. Denoting by 2α, N, i, d, r, order, class, number of inflexions (including cusps of the second kind), double-points, cusps, we have

$$i = 6\alpha(2\alpha - 2) - 6d - 8r,$$

an even number of inflexions. For the class N we have

$$N = 2\alpha(2\alpha - 1) - 2d - 3r,$$

and

$$N + r = 2[\alpha(2\alpha - 1) - d - r],$$

which is even. Hence as found before, the number of cusps and tangents from a point to the curve is even.*

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* For other topological properties of closed curves see G. Landsberg: "Beitrag zur Topologie geschlossener Kurven mit Knotenpunkten und zur Kroneckerschen Charakteristikentheorie," Math. Annalen, vol. 70 (1911), pp. 563–579; in the same publication A. Kneser: "Einige allgemeine Sätze über die einfachsten Gestalten ebener Kurven," vol. 41 (1893), pp. 347–376. v. Staudt in his Geometrie der Lage, p. 113 (1847), gave a purely intuitional proof of the theorem that every closed curve has an even number of inflexions (defined in the sense of v. Staudt).

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